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ON THE GRAM DETERMINANT

By C. F. MOPPERT (*University of Melbourne*)

[Received 2 December 1957]

Introduction

In a recent note in this *Journal* W. N. Everitt has published some useful inequalities concerning Gram determinants defined over the integrable-square function spaces [see (1)]. In this note I state and prove a simple geometrical theorem in Hilbert space and then use this result to generalize the inequalities, given by Everitt in (1), to Gram determinants defined over an arbitrary Hilbert space. By a suitable choice of a projection operator on the Hilbert space of integrable-square functions the inequalities in (1) then follow as a special case of this more general result.

1. A geometrical theorem

Let x, y, a, b, c, and f, with or without subscripts, denote vectors from a Hilbert space $\mathscr{H}$. The following definitions will hold:

(i) $V(f_1,f_2,...,f_n)$ will denote the linear manifold in $\mathscr{H}$ spanned by the vectors $f_1, f_2,..., f_n$;

(ii) $\|f-M\|$ will denote the 'distance' of the vector f from a linear manifold M, i.e. $\|f-M\| = \inf\{\|f-x\|; x \in M\}$;

(iii) given f and M as in (ii), then f_M will denote that vector of M such that $\|f-M\| = \|f-f_M\|$. It is known [see (2) 23] that f_M is unique, $(f-f_M, x) = 0$ for all x in M and conversely, if $f_1 \in M$ with $(f-f_1, x) = 0$ for all x in M, then $f_1 = f_M$.

THEOREM 1. *Given the vectors a, a_1, a_2,..., a_n and a projection P let $b = Pa$, $b_r = Pa_r$ $(r = 1, 2, ..., n)$, then*

$$\|a-A\| \geqslant \|b-B\|, \tag{1.1}$$

where $A = V(a_1, a_2,..., a_n)$ and $B = V(b_1, b_2,..., b_n)$. Equality holds if and only if

$$Pa-Pa_A = a-a_A. \tag{1.2}$$

Since P is a projection, we have

$$\|a-A\| = \|a-a_A\| \geqslant \|Pa-Pa_A\| \tag{1.3}$$

and, since also $a_A \in A$, we have $Pa_A \in B$; thus, with (ii) above,

$$\|Pa-Pa_A\| = \|b-Pa_A\| \geqslant \|b-B\|, \tag{1.4}$$

and the inequality (1.1) follows from (1.3) and (1.4).

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There is equality in (1.3), since P is a projection, if and only if (1.2) is satisfied.

To discuss equality in (1.4) suppose to begin with that (1.2) is satisfied. Let $y \in B$; then there is a vector x in A such that $y = Px$ and

$$\begin{aligned}(b-Pa_A, y) &= (Pa-Pa_A, Px)\\ &= (Pa-Pa_A, x), \quad \text{since } P \text{ is a projection},\\ &= (a-a_A, x), \quad \text{from (1.2)},\\ &= 0, \quad \text{from (iii) above.}\end{aligned}$$

Thus $(b-Pa_A, y) = 0$ for all y in B, and so $Pa_A = b_B$,

i.e. $$\|b-Pa_A\| = \|b-b_B\| = \|b-B\|,$$

and equality holds in (1.4).

On the other hand, if equality holds in (1.4), we have $Pa_A = b_B$. Let $a-a_A = Pa-Pa_A+c$, so that $c \in A$. Then, since for all x in A we have $(a-a_A, x) = 0$, it follows that

$$(Pa-Pa_A, x)+(c, x) = 0. \tag{1.5}$$

Now $$\begin{aligned}(Pa-Pa_A, x) &= (Pa-Pa_A, Px)\\ &= (b-b_B, Px)\\ &= 0, \quad \text{since } Px \in B.\end{aligned}$$

Thus, from (1.5), we have $(c, x) = 0$ for all x in A, and consequently c is the null vector of $\mathscr{H}$, i.e. $a-a_A = Pa-Pa_A$.

Thus there is equality in (1.4) if and only if (1.2) is satisfied, and this completes the proof of the theorem.

2. Gram determinants

Let $G(f_1, f_2, \ldots, f_n)$ denote the Gram determinant of the vectors $f_1, f_2, \ldots, f_n$ of $\mathscr{H}$, i.e.

$$G(f_1, f_2, \ldots, f_n) \equiv \det[(f_r, f_s)] \quad (1 \leqslant r, s \leqslant n).$$

It is known [see (3) 13] that

$$G(f_1, f_2, \ldots, f_n) = G(f_2, \ldots, f_n)\|f_1 - V(f_2, \ldots, f_n)\|^2, \tag{2.1}$$

and from this, or otherwise, it follows that G is real-valued and non-negative.

THEOREM 2. *Given the vectors $a_1, a_2, \ldots, a_n$ and the projection P with $b_r = Pa_r$ $(r = 1, 2, \ldots, n)$, then*

$$G(a_1, a_2, \ldots, a_n) \geqslant G(b_1, b_2, \ldots, b_n) \tag{2.2}$$

with equality if and only if $a_r = b_r$ $(r = 1, 2, \ldots, n)$.

The result clearly is true when $n = 1$ since P is a projection. Suppose then that the result is true for some n with $n \geqslant 1$. Then†

$$\begin{aligned} G(a_1, a_2, \ldots, a_n, a_{n+1}) &= \|a_{n+1} - A\| G(a_1, a_2, \ldots, a_n), \quad \text{from (2.1)}, \\ &\geqslant \|a_{n+1} - A\| G(b_1, b_2, \ldots, b_n), \quad \text{from (2.2)}, \\ &\geqslant \|b_{n+1} - B\| G(b_1, b_2, \ldots, b_n), \quad \text{from (1.1)}, \\ &= G(b_1, b_2, \ldots, b_n, b_{n+1}), \end{aligned}$$

and (2.2) follows for the case $n+1$. There is equality if and only if $a_r = b_r$ $(r = 1, 2, \ldots, n)$ and also $a_{n+1} - a_A = Pa_{n+1} - Pa_A$.

But by the induction hypothesis $a_A \in A = B$ so that $Pa_A = a_A$. This completes the proof.

THEOREM 3. *With the notation of Theorem* 2

$$\frac{G(b_1, b_2, \ldots, b_n)}{G(b_1, b_2, \ldots, b_n, b_{n+1})} \geqslant \frac{G(a_1, a_2, \ldots, a_n)}{G(a_1, a_2, \ldots, a_n, a_{n+1})}. \tag{2.3}$$

Proof. This follows directly from Theorem 1 and the relationship (2.1).

Note. This result (2.3) has to be suitably interpreted if one or more of the G-factors is zero; this can be done consistently if Theorem 2 is used where necessary.

3. The function space L^2

Let E_1 and E_2 denote measurable linear sets satisfying $E_1 \subseteq E_2$. The function spaces L_1^2 on E_1 and L_2^2 on E_2 are defined in the usual manner. The notation $\| \ \|_r$ and $(\ , \ )_r$ $(r = 1, 2)$ will be used for the norm and scalar product in these spaces.

Let $\phi(x)$ denote the characteristic function of the set E_1, i.e. $\phi(x) = 1$ for x in E_1 and zero otherwise.

To any function $f(x)$ of L_2^2 we can associate in a unique way a function $g(x)$ of L_1^2 by putting $g(x) = f(x)$ when $x \in E_1$. On the other hand, putting $f^*(x) = \phi(x) f(x)$, we have

$$f^*(x) \in L_2^2 \quad \text{and} \quad \|f^*\|_2 = \|g\|_1.$$

Furthermore, if f_1, f_2, g_1, g_2, f_1^*, f_2^* have the corresponding meaning then

$$(f_1^*, f_2^*)_2 = (g_1, g_2)_1.$$

It is clear that the transformation on L_2^2 defined by $f^* \equiv \phi f$ is a

† A and B have the same meaning as in Theorem 1.

projection P (say), i.e., if $Pf \equiv f^*$, then

$$P^2 = P \quad \text{and} \quad (Pf_1, f_2)_2 = (f_1, Pf_2)_2.$$

With this interpretation of P and the space $\mathscr{H}$ it follows that Theorems 1 and 2 of (**1**) now appear as special cases, respectively, of Theorems 2 and 3 of this note.

I am indebted to the referee for many improvements: in particular for completing the proof of Theorem 1.

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2. P. R. Halmos, *Introduction to Hilbert Space* (New York, 1951).

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SOME REMARKS ON VARIETIES OF GROUPS

By GRAHAM HIGMAN (*Oxford*)

[Received 19 December 1957]

1. Introduction

In this note three results are proved about varieties of groups. They have little in common, beyond the fact that they are foreshadowed in, or answer questions raised by Hanna Neumann's paper (**5**). We begin by recalling the main facts.

By definition, a *variety of groups* is a class of groups closed under the operations of taking subgroups, homomorphic images, and (strong or unrestricted) direct products. By a well-known theorem of G. Birkhoff (**1**) this is equivalent to saying that a variety is the class of all groups satisfying a given set of identical relations. Evidently we may take this set to be closed: that is, to include all its consequences. Then there is a one-to-one correspondence between varieties and closed sets of identical relations; the variety contains all groups which satisfy the relations, just as the set contains all relations satisfied by all groups of the variety. Obviously, if the varieties $\mathfrak{U}$, $\mathfrak{V}$ correspond to the sets U, V of identical relations, $\mathfrak{U} \subset \mathfrak{V}$ if and only if $V \subset U$.

An apparently difficult question here is whether or not there exist properly infinite descending chains of varieties; or what comes to the same thing, ascending chains of closed sets of identical relations. Since an identical relation follows from a given set of identical relations only if it follows from some finite subset of it, an equivalent question is whether every closed set of identical relations is implied by a finite set (in which case it is implied by a single relation). On the other hand, it is easy to construct properly infinite ascending chains of varieties, or descending chains of sets of identical relations. Here, however, we cannot deduce that there are varieties not generated† by a finite set of groups because, owing to the direct-product condition, a group may belong to the variety generated by an infinite set of groups, without belonging to the variety generated by any of its finite subsets. And, in fact, every variety is generated by a single group, namely, its relatively free group on a countable infinity of generators.

† The variety *generated* by a set of groups is the least variety containing them all.

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Two of the results here are connected with this question of the existence of a finite basis for a set of identical relations. First, we prove that a basis for the relations of the variety consisting of all groups all of whose two-generator subgroups are metabelian is provided by the identical relation $((x, y), (x^{-1}, y)) = 1$. Secondly, call the variety $\mathfrak{U}$ *nilpotent* if every group in $\mathfrak{U}$ is nilpotent, in which case there is a common bound for the classes of the groups in $\mathfrak{U}$. Then R. C. Lyndon (**3**) has shown that the identical relations of a nilpotent variety are finitely based. We generalize this by showing that, if $\mathfrak{U} \circ \mathfrak{V}$ denotes the variety of groups with a normal subgroup in $\mathfrak{U}$ with factor group in $\mathfrak{V}$, then the relations of $\mathfrak{U} \circ \mathfrak{V}$ are finitely based provided that $\mathfrak{U}$ is nilpotent and the relations of $\mathfrak{V}$ are finitely based.

The third result concerns a slightly different problem. From the second characterization of varieties it is clear that a variety is determined by the finitely generated groups belonging to it, in the sense that no other variety contains precisely the same finitely generated groups. One of the questions studied by Mrs. Neumann is whether in fact a variety is necessarily determined, in a similar sense, by its n-generator groups for some fixed finite n. I show that this is not so. More precisely, following Mrs. Neumann, but in a different notation, one may consider, for any variety $\mathfrak{V}$, the variety $\mathfrak{V}_n$ generated by the n-generator groups of $\mathfrak{V}$, and the variety $\mathfrak{V}^n$ of groups satisfying those identical relations of $\mathfrak{V}$ which involve at most n variables. ($\mathfrak{V}^n$ is the variety of all groups whose n-generator subgroups belong to $\mathfrak{V}$.) Then $\mathfrak{V}_n \subset \mathfrak{V} \subset \mathfrak{V}^n$, and $\mathfrak{V}_n$ and $\mathfrak{V}^n$ are respectively the smallest and largest varieties having the same n-generator groups as $\mathfrak{V}$. Moreover

$$\mathfrak{V}_1 \subset \mathfrak{V}_2 \subset \mathfrak{V}_3 \subset \ldots \subset \mathfrak{V}_n \subset \ldots$$

is an ascending chain of varieties whose union generates $\mathfrak{V}$, and

$$\mathfrak{V}^1 \supset \mathfrak{V}^2 \supset \mathfrak{V}^3 \supset \ldots \supset \mathfrak{V}^n \supset \ldots$$

a descending chain whose intersection is $\mathfrak{V}$. Thus $\mathfrak{V}$ is determined by its n-generator groups, for some n, if neither of these chains is properly infinite. For the second chain we have a special case of the problem of a finite base previously mentioned, which is not obviously easier than the general case. However, I show that the first chain is properly infinite if $\mathfrak{V} = \mathfrak{A} \circ \mathfrak{B}$, where $\mathfrak{A}$ is the variety of abelian groups of prime exponent p, and $\mathfrak{B}$ is any variety, containing at least one non-abelian group, in which finitely generated groups are finite and of order prime to p.

2. Groups whose two-generator subgroups are metabelian

THEOREM 2.1. *A group satisfies the identical relation* $((x,y),(x^{-1},y))=1$ *if and only if all its two-generator subgroups are metabelian.*

Here, of course, (x,y) is the commutator $x^{-1}y^{-1}xy$; and the non-trivial part of the theorem is that the given relation implies that all two-generator subgroups are metabelian. For convenience, we abbreviate (x^i,y^j) to $c_{i,j}$, and write $x \rightleftarrows y$ for 'x commutes with y'. Thus the given relation is

$$c_{1,1} \rightleftarrows c_{-1,1}. \tag{2.1}$$

If here we substitute $y^a x$ for x, where a is an integer, positive or negative, we obtain

$$c_{1,1} \rightleftarrows c_{-1,a}^{-1} c_{-1,-a+1}.$$

This yields, by an obvious induction on $|a|$, the relation

$$c_{1,1} \rightleftarrows c_{-1,a} \tag{2.2}$$

for all integers a. Now make the further substitution of $y^{-1}x^{-1}y$ for y. The result is

$$c_{1,1}c_{-1,1} \rightleftarrows c_{-1,1}c_{-a-1,1}^{-1}c_{-a,1}.$$

But $c_{1,1}c_{-1,1}$ commutes with $c_{-1,1}$ by (2.1), so that again induction on $|a|$ gives

$$c_{1,1}c_{-1,1} \rightleftarrows c_{a,1}. \tag{2.3}$$

We now prove by induction on $|a|$ the relation

$$c_{1,1} \rightleftarrows c_{a,1}, \tag{2.4}$$

which is trivial, or given by (2.1), if $|a| \leqslant 1$. Assume then that $a > 1$, and assume that $c_{1,1}$ commutes with $c_{i,1}$ if $|i| < a$. Then, in particular,

$$c_{1,1} \rightleftarrows c_{1-a,1}.$$

In this relation we substitute xyx^{-1} for y. We obtain

$$c_{-1,1}^{-1} \rightleftarrows c_{-a,1}c_{-1,1}^{-1},$$

which is to say

$$c_{-1,1} \rightleftarrows c_{-a,1}. \tag{2.5}$$

Comparison of (2.5) with (2.3) shows that $c_{1,1}$ commutes with $c_{-a,1}$; substitution of x^{-1} for x in (2.5) shows that $c_{1,1}$ commutes with $c_{a,1}$. This completes the inductive proof of (2.4). If in that formula we interchange x and y, we get

$$c_{1,1} \rightleftarrows c_{1,a}. \tag{2.6}$$

Finally, we prove, by induction on $|i|$, that, for any integers i, j

$$c_{1,1} \rightleftarrows c_{i,j}. \tag{2.7}$$

This is trivial, or contained in (2.2) and (2.6), if $|i| \leqslant 1$. Assume then that $i > 1$, and that $c_{1,1}$ commutes with $c_{a,j}$ if $|a| < i$. Then, inter-

changing x and y, we see that $c_{1,1}$ also commutes with $c_{j,a}$. In particular, if $|a| < i$,

$$c_{1,1} \overset{\rightarrow}{\leftarrow} c_{i,a}.$$

In this relation we substitute $y^r x$ for x. The result is

$$c_{1,1} \overset{\rightarrow}{\leftarrow} d_{r,i,a},$$

where

$$d_{r,i,a} = c_{1,r}c_{2,r}^{-1}c_{2,2r}c_{3,2r}^{-1}\cdots c_{i-1,(i-1)r}c_{i,(i-1)r}^{-1}c_{i,(i-1)r+a}c_{i-1,(i-1)r+a}^{-1}\cdots \times$$
$$\times c_{2,r+a}c_{1,r+a}^{-1}c_{1,a}.$$

Thus, remembering that we are assuming that $c_{1,1}$ commutes with $c_{k,j}$ if $|k| < i$, we have

$$c_{1,1} \overset{\rightarrow}{\leftarrow} c_{i,(i-1)r}^{-1}c_{i,(i-1)r+a}$$

for any integers r and a such that $|a| < i$. It follows that, if $c_{1,1}$ commutes with $c_{i,j}$ for one value of j in the range

$$(i-1)(r-1) \leqslant j \leqslant (i-1)(r+1),$$

it commutes with $c_{i,j}$ for all such j. Since these intervals, for successive values of r, overlap, and $c_{1,1}$ commutes with $c_{i,1}$ by (2.4), $c_{1,1}$ commutes with $c_{i,j}$ for all j. A similar argument shows that $c_{1,1}$ commutes with $c_{-i,j}$, only starting from the relations

$$c_{1,1} \overset{\rightarrow}{\leftarrow} c_{-i,a}.$$

This completes the inductive proof of (2.7).

The theorem now follows. For the derived group of $\{x, y\}$ is generated, on the one hand, by the set of all $c_{i,j}$, and on the other by the set of all conjugates of $c_{1,1}$. By (2.7) the centre of the derived group contains $c_{1,1}$, and so every conjugate of $c_{1,1}$, and hence it is the whole derived group: that is, $\{x, y\}$ is metabelian, as required.

3. Composition of nilpotent varieties

If $x_0, x_1, x_2, \ldots$ are group variables, we define h_n inductively by setting $h_0 = x_0$, and $h_{i+1} = (h_i, x_{i+1})$. Then a group, or a variety, is nilpotent of class n if $h_n = 1$ is one of its identical relations. Generally, we denote by $H_n(G)$ the subgroup of the group G generated by values of h_n for x_i in G. It is well known that $H_n(G)$ is the normal closure of the subgroup generated by the values of h_n for x_i in some generating set of G, so that, in particular, to verify that G is nilpotent of class n one only need check that $h_n = 1$ for x_i in a generating set.

We recall that, if $\mathfrak{A}$ and $\mathfrak{B}$ are varieties of groups, $\mathfrak{A} \circ \mathfrak{B}$ is the variety of groups with a normal subgroup in $\mathfrak{A}$ whose factor group is in $\mathfrak{B}$. Our object is to prove the following theorem:

THEOREM 3.1. *The identical relations of $\mathfrak{A} \circ \mathfrak{B}$ are finitely based so long as $\mathfrak{A}$ is nilpotent and the identical relations of $\mathfrak{B}$ are finitely based.*

We begin with some arguments similar to ones due to Lyndon (3) and Tobin [(6) especially Lemma 5.1]. Let F be the free group freely generated by $x_0, x_1, x_2, \ldots$. Let π_i be the endomorphism of F which maps x_i on 1, and x_j on x_j for $j \neq i$. Let F_i be the kernel of π_i, and, for any set J of integers, let F_J be the intersection of the groups F_i for i in J.

LEMMA 3.2. *If the number of elements in J is n, $F_J \subset H_{n-1}(F)$.*

This lemma is a consequence of a more precise result. The notion of a commutator in the generators x_i is defined inductively; x_i and x_i^{-1} are commutators for all i; if y and z are commutators, so is (y, z); and nothing is a commutator unless it is required by these rules. The commutator u involves x_i if it is either x_i or x_i^{-1}, or (y, z), where y or z involves x_i. Then a commutator which involves n distinct x_i certainly belongs to $H_{n-1}(F)$, so that it is sufficient to prove the lemma:

LEMMA 3.3. *Every element of F_J can be written as a product of commutators, each of which involves x_i for each i in J.*

This is trivial if J is empty; so we prove it by induction on n. Then we may suppose w in F_J written as

$$w = u_1 u_2 u_3 \ldots u_r,$$

where each commutator u_i contains each x_j (j in J), with the possible exception of one fixed x_a. Moreover, at the cost of introducing extra commutators which contain all x_j, we can suppose that the u_i not containing x_a form the terminal segment $u_{s+1} \ldots u_r$. But then

$$1 = w\pi_a = u_{s+1} \ldots u_r,$$

so that

$$w = u_1 u_2 \ldots u_s$$

is the required expression.

Next, we write $y(1-\pi_i)$ for $y(y\pi_i)^{-1}$, and note that, for any y,

$$y(1-\pi_i)\pi_i = 1, \qquad y(1-\pi_i)\pi_j = y\pi_j(1-\pi_i).$$

Evidently, then,

$$y(1-\pi_0)(1-\pi_1)\ldots(1-\pi_n)$$

belongs to F_{I_n}, where $I_n = \{0, 1, \ldots, n\}$, and so, by Lemma 3.2, to $H_n(F)$. This can be expressed in the following terms.

LEMMA 3.4. *For any n, any element y of F can be written*

$$y = uv_1 v_2 \ldots v_r,$$

where u belongs to $H_n(F)$, and each v_i is $y^{\pm 1}\pi_{a_1}\pi_{a_2}\ldots\pi_{a_s}$ for some non-empty sequence $a_1, a_2, \ldots, a_s$ of integers between 0 and n.

It is a corollary of this lemma that any identical relation is equivalent

to a set of identical relations each involving at most n variables, together with a relation implied by $h_n = 1$. The case $n = 1$ of this is due to B. H. Neumann (**4**), the general case to Lyndon (**3**), who used it to show that the identical relations of a nilpotent group are finitely based. Here we are interested in a different application.

LEMMA 3.5. *Let G be a nilpotent group of class n, generated by elements g_λ, λ in Λ. Then G satisfies the identical relation $w(x_1, x_2, ..., x_r) = 1$ provided that $w(h_1, ..., h_r) = 1$ whenever $h_1, ..., h_r$ are words in the g_λ and g_λ^{-1} of lengths $l_1, ..., l_r$, with $\sum l_i \leqslant n$.*

We have, of course, to show that $w(h_1, ..., h_r) = 1$, when the h_i are as in the lemma, but without the length restriction. We assume that h_i is of length l_i, where $l = \sum l_i$ is greater than n, and use induction on l. Now $w(h_1, ..., h_r)$ is obtained from

$$y = w(x_0 x_1 \dots x_{l_1 - 1};\ x_{l_1} \dots;\ \dots;\ \dots x_{l-1})$$

by a homomorphism of F into G in which each x_i maps into some $g_\lambda^{\pm 1}$. We express y in the form indicated in Lemma 3.4. Because l is greater than n, each of $x_0, ..., x_n$ actually occurs in y, and the hypothesis of the induction therefore ensures that v_i is mapped under the homomorphism into 1. But u is mapped into 1 because G is nilpotent of class n. Hence y is mapped into 1, giving the result.

Theorem 3.1 is an almost immediate consequence. Let $\mathfrak{A}$ be defined by the identical relation $a(x_1, ..., x_r) = 1$, $\mathfrak{B}$ by the relation $b(...) = 1$. Then in general we have as defining identical relations for $\mathfrak{A} \circ \mathfrak{B}$ the relations

$$a(y_1, ..., y_r) = 1, \tag{3.1}$$

where

$$y_i = b_{i1}^{\epsilon_{i1}} b_{i2}^{\epsilon_{i2}} \dots b_{is_i}^{\epsilon_{is_i}},$$

where $\epsilon_{ij} = \pm 1$, and the b_{ij} are copies of b with non-overlapping variable sets. In general it is not clear that we can limit the values of s_i, to obtain an equivalent finite set. If, however, $\mathfrak{A}$ is nilpotent of class n, we can include

$$(b_0, b_1, ..., b_n) = 1$$

among our identical relations, and then, by Lemma 3.5, we need take in addition only those (3.1) for which $\sum s_i \leqslant n$, a finite set.

In view of the results of the next section, it is perhaps worth while to add the following consequences of Lemma 3.5.

THEOREM 3.6. *A variety which is nilpotent of class n is generated by its n-generator groups.*

For, if not, the n-generator groups generate a proper subvariety, and therefore satisfy an identical relation not satisfied by all groups of the

variety. But it is a corollary of Lemma 3.5 [cf. also Lyndon (3)] that this relation can be taken to involve only n variables, and then, if it holds for the n-generator groups of the variety, it holds for all, a contradiction.

THEOREM 3.7. *A variety which is nilpotent of class n is determined by its $(n+1)$-generator groups.*

Suppose that another variety had the same $(n+1)$-generator groups. Then it also would be nilpotent of class n since the $(n+1)$-generator groups are enough to ensure this. Thus both varieties, by Theorem 3.6, are generated by their n-generator groups, which are the same. Hence the two varieties are the same.

4. Varieties not determined by their n-generator groups for any n

Throughout this section we write $\mathfrak{V} = \mathfrak{A} \circ \mathfrak{B}$, where $\mathfrak{A}$ is the variety of abelian groups of exponent p, for some prime p, and $\mathfrak{B}$ is a variety, containing at least one non-abelian group, in which every finitely generated group is finite and of order prime to p. It may be noted that there is no shortage of such varieties $\mathfrak{B}$; by Lemma 4.3 below, or by B. H. Neumann (4), we may take $\mathfrak{B}$ to be the variety generated by any non-abelian group of finite order prime to p.

If $\mathfrak{X}$ is a class of groups, the variety generated by $\mathfrak{X}$ consists, of course, of the groups obtained from them by applying as often as necessary the operations of taking subgroups, homomorphic images, and direct products. But in fact, if we do it in the right order, we need perform each of these operations essentially only once.

LEMMA 4.1. *A group G which belongs to the variety generated by $\mathfrak{X}$ is a homomorphic image of a subgroup of a direct product of factors isomorphic to groups in $\mathfrak{X}$.*

For it is easy to verify that the class of groups described is a variety.

LEMMA 4.2. *Suppose that either* (i) *$\mathfrak{X}$ is a finite set of finite groups and G is finitely generated; or* (ii) *G is finite. Then G is isomorphic to a subgroup of a direct product of factors isomorphic to groups in $\mathfrak{X}$ only if it is isomorphic to a subgroup of a direct product of a finite number of such factors.*

If G is isomorphic to a subgroup of a direct product of factors isomorphic to groups in $\mathfrak{X}$, then it possesses a family of normal subgroups

intersecting in the identity whose factor groups are isomorphic to subgroups of groups in $\mathfrak{X}$. But under either hypothesis of the lemma such a family is finite, and the lemma follows.

LEMMA 4.3. *Suppose that either* (i) $\mathfrak{X}$ *is a finite set of finite groups; or* (ii) *all groups in* $\mathfrak{X}$ *belong to a fixed variety* $\mathfrak{U}$ *in which finitely generated groups are finite. Then a finitely generated group* G *which belongs to the variety generated by* $\mathfrak{X}$ *is a homomorphic image of a direct product of a finite number of factors isomorphic to groups in* $\mathfrak{X}$.

Lemma 4.1 assures us that G is a homomorphic image of a subgroup of a direct product of factors isomorphic to groups in $\mathfrak{X}$. This subgroup can evidently be taken to be finitely generated, and hence, under the second hypothesis of the lemma, to be finite. The result now follows from Lemma 4.2. It is, of course, a corollary that the first hypothesis implies the second.

We show next that Lemma 4.3 applies if $\mathfrak{X}$ is a set of groups in $\mathfrak{V} = \mathfrak{A} \circ \mathfrak{B}$.

LEMMA 4.4. *Finitely generated groups in* $\mathfrak{V}$ *are finite.*

If G belongs to $\mathfrak{V}$, it has a normal subgroup N in $\mathfrak{A}$ such that G/N belongs to $\mathfrak{B}$. If G is finitely generated, so is G/N, so that N is of finite index in G. Hence N is also finitely generated, and so of finite order. The lemma follows. [Cf. (**2**) Lemma 4.2.2.]

Our final preparatory lemma states that in determining the finite groups in a variety we may confine our attention to groups with only one minimal subgroup. [Cf. (**2**) Lemma 1.2.1.]

LEMMA 4.5. *If* $\mathfrak{U}$ *and* $\mathfrak{V}$ *are any two varieties of groups, every finite group in* $\mathfrak{U}$ *belongs also to* $\mathfrak{V}$, *provided that this is true for every finite group with only one minimal normal subgroup.*

This is an immediate consequence of the fact that, if a group G has two distinct minimal normal subgroups H and K, they intersect in the identity, and G is therefore a subgroup of the direct product of G/H and G/K.

We turn now to the special case of the variety $\mathfrak{V} = \mathfrak{A} \circ \mathfrak{B}$. If G is a finite group belonging to $\mathfrak{V}$, G has a normal subgroup N in $\mathfrak{A}$ with factor group G/N in $\mathfrak{B}$. Then N is an elementary abelian p-group and can therefore be thought of as the additive group of a vector space over the field F_p of p elements. Then the automorphisms of N induced by transformation by elements of G are linear transformations of this vector space, and constitute a representation of G/N over F_p. Again,

by Schur's theorem [Zassenhaus (7) 125], N has a complement in G, so that the group G/N of $\mathfrak{B}$ and its representation over F_p determine, as well as being determined by, G. Furthermore, the representation is completely reducible, so that N is a direct product of minimal normal subgroups of G, and, if N_1 is any normal subgroup of G in N, then N is the direct product of N_1 and N_2, where N_2 also is normal in G.

Our aim is to determine the varieties that contain $\mathfrak{B}$ and are contained in $\mathfrak{V}$. Evidently, from Lemma 4.5, the finite groups in $\mathfrak{V}$ with only one minimal normal subgroup will be important here. If G is such a group, then either N is 1, in which case G belongs to $\mathfrak{B}$ and is from the present point of view uninteresting, or N is the minimal normal subgroup of G. In this case, the corresponding representation of G/N is evidently irreducible; and it is moreover [(2) Lemma 1.2.3] faithful; that is, G is determined by an irreducible linear group over F_p in $\mathfrak{B}$. If X, Y are two such groups, we say that Y is a *factor* of X if there is a subgroup X_0 of X and a subspace V_0 of the space V on which X operates such that V_0 admits X_0, and Y is isomorphic as linear group to the restriction of X_0 to V_0. A class $\mathfrak{X}$ of irreducible linear groups will be called *closed* if it contains with X also every irreducible factor of X.

THEOREM 4.6. *Let $\mathfrak{X}$ be a closed class of irreducible linear groups over F_p belonging to $\mathfrak{B}$. Then there is a variety $\mathfrak{U}(\mathfrak{X})$ containing $\mathfrak{B}$ and contained in $\mathfrak{V}$ such that the finite group G in $\mathfrak{V}$ belongs to it if and only if, for each minimal normal p-subgroup of G, the group of automorphisms induced in it by transformation by elements of G, considered as a linear group, belongs to $\mathfrak{X}$. Moreover every variety containing $\mathfrak{B}$ and contained in $\mathfrak{V}$ is of this form for some closed class $\mathfrak{X}$.*

Let $\mathfrak{U}_0(\mathfrak{X})$ be the class of finite groups described in the theorem and let $\mathfrak{U}(\mathfrak{X})$ be the variety generated by them. In order to verify that a finite group G belongs to $\mathfrak{U}(\mathfrak{X})$ only if it belongs to $\mathfrak{U}_0(\mathfrak{X})$ it suffices, by Lemma 4.3, to show that subgroups and homomorphic images of groups in $\mathfrak{U}_0(\mathfrak{X})$ belong to $\mathfrak{U}_0(\mathfrak{X})$, and that the direct product of two groups in $\mathfrak{U}_0(\mathfrak{X})$ belongs to $\mathfrak{U}_0(\mathfrak{X})$. Moreover, in applying the condition, to check that a group G belongs to $\mathfrak{U}_0(\mathfrak{X})$, we evidently need consider only those minimal normal p-subgroups occurring in one fixed decomposition of the maximal normal p-subgroup of G into a direct product of minimal normal subgroups. For, by complete reducibility, every minimal normal p-subgroup occurs in some such decomposition, and, by the Krull–Schmidt theorem, the factors in any two such decompositions, in some

order, are isomorphic, as groups with operators in G, so that *a fortiori* the linear groups induced in them are isomorphic.

Now let G belong to $\mathfrak{U}_0(\mathfrak{X})$, and let us seek to prove that the subgroup H of G also belongs to $\mathfrak{U}_0(\mathfrak{X})$. Since we can pass from G to H via NH, where N is the maximal normal p-subgroup of G, there is no loss of generality in supposing the index of H in G to be either prime to p or a power of p. If it is prime to p, N is contained in H. We can decompose N into a direct product of minimal normal subgroups of H by first decomposing it into a product of minimal normal subgroups of G, and then further decomposing these factors as necessary. Then, by definition, the linear group induced by H in one of these minimal normal p-subgroups is a factor of the linear group induced by G in the minimal normal subgroup of G containing it, and so also belongs to $\mathfrak{X}$. If, however, the index of H is a power of p, then $G = NH$. Since N centralizes any p-subgroup of G, a minimal normal p-subgroup of H is normal also in G, and H induces in it the same linear group that G does, which is certainly in $\mathfrak{X}$.

Next, we must prove that the factor group G/K belongs to $\mathfrak{U}_0(\mathfrak{X})$, for any normal subgroup K of G. Here, using $G/N \cap K$ as intermediary, we see that we may suppose that K is of order either a power of p or prime to p. If K has order a power of p, K is contained in N, and, by complete reducibility, a minimal normal p-subgroup of G/K is $N_1 K/K$, where N_1 is a minimal normal p-subgroup of G, and the product $N_1 K$ is direct. If the order of K is prime to p, the same conclusion is immediate since K and N then commute element-wise. In either case, the linear group induced by G/K in $N_1 K/K$ is isomorphic to that induced by G in N_1 and so belongs to $\mathfrak{X}$.

Lastly, if G_i belongs to $\mathfrak{U}_0(\mathfrak{X})$ $(i = 1, 2)$ and has maximal normal p-subgroup N_i, then the maximal normal p-subgroup of the direct product $G_1 G_2$ is the direct product $N_1 N_2$. Thus we need consider only minimal normal p-subgroups of $G_1 G_2$ contained either in N_1 or in N_2. The linear groups induced in these are plainly those induced by G_1 or G_2, and so belong to $\mathfrak{X}$. This proves the first statement of the theorem.

To prove the second statement, let $\mathfrak{U}$ be a variety containing $\mathfrak{B}$ and contained in $\mathfrak{V}$. Let $\mathfrak{X}$ be the class of those irreducible linear groups over F_p belonging to $\mathfrak{B}$ for which the corresponding group in $\mathfrak{V}$ belongs to $\mathfrak{U}$. $\mathfrak{X}$ is evidently closed, and we shall prove that $\mathfrak{U} = \mathfrak{U}(\mathfrak{X})$. Since every variety is determined by its finitely generated groups, a subvariety of $\mathfrak{B}$ is determined by its finite groups. Thus by Lemma 4.5 it is sufficient to show that the finite groups with only one minimal normal

subgroup in $\mathfrak{U}$ and $\mathfrak{U}(\mathfrak{X})$ are the same. But, as was noted before the statement of Theorem 4.6, a group in $\mathfrak{B}$ with only one minimal normal subgroup is either in $\mathfrak{B}$, and so in both $\mathfrak{U}$ and $\mathfrak{U}(\mathfrak{X})$, or is determined by an irreducible linear group over F_p, in which case it belongs alike to $\mathfrak{U}$ and to $\mathfrak{U}(\mathfrak{X})$ if and only if the irreducible group belongs to $\mathfrak{X}$.

Remark 4.7. It should be pointed out that we have not yet used the hypothesis that $\mathfrak{B}$ contains a non-abelian group, and Theorem 4.6 is therefore valid without that hypothesis. Theorem 4.8, which follows, is not; see example 4.9 below.

THEOREM 4.8. *The variety $\mathfrak{B}$ is not generated by any finite set of finite groups; in particular, it is not generated by its n-generator groups for any finite n.*

We note first that the second statement is indeed a particular case of the first; for, if $\mathfrak{B}$ were generated by its n-generator groups, it would be generated by its relatively free n-generator group, which is finite.

Now, if r is an integer, those irreducible linear groups over F_p belonging to $\mathfrak{B}$ which operate on spaces of dimension at most r over F_p evidently form a closed class $\mathfrak{X}_r$, and any finite set of finite groups in $\mathfrak{B}$ will belong to $\mathfrak{U}(\mathfrak{X}_r)$ for r large enough. It is enough, then, to show that for no r does $\mathfrak{X}_r$ contain all irreducible linear groups over F_p belonging to $\mathfrak{B}$. Now there is a non-abelian group G in $\mathfrak{B}$, which has therefore an absolutely irreducible representation of degree at least 2 over an extension field of F_p. The repeated Kronecker product of this representation with itself gives an absolutely irreducible representation of G^r of degree at least 2^r. This is contained as absolutely irreducible constituent in at least one irreducible representation of G^r over F_p, which also must be of degree at least 2^r. This is an irreducible linear group over F_p belonging to $\mathfrak{B}$ not in $\mathfrak{X}_r$.

We conclude with two examples. It is convenient to prefix them with a remark. It is clear that the correspondence between closed classes $\mathfrak{X}$ and varieties $\mathfrak{U}(\mathfrak{X})$ is one-to-one with the trivial exception that the variety $\mathfrak{B}$ itself corresponds both to the empty class and to the class containing only the unit group operating on a vector space of dimension zero, if this is admitted as an irreducible group. For convenience, we insist below that an irreducible group must operate on a space not consisting of zero alone, and so resolve the ambiguity.

Example 4.9. We write ${}_n\mathfrak{A}$ for the variety of abelian groups of exponent n, so that $\mathfrak{A}$ above is ${}_p\mathfrak{A}$, and we take $\mathfrak{B} = {}_n\mathfrak{A}$ for some n

prime to p. Then Theorem 4.6 applies in virtue of remark 4.7. As defining identical relations for $\mathfrak{B}$ one may take

$$x^{np} = 1, \quad (x, y)^p = 1, \quad (x^n, y^n) = 1.$$

For n is prime to p, so that the third equation implies that the elements of order p generate an abelian normal subgroup, and the first two imply that its factor group is of exponent n and abelian. The irreducible linear groups in $\mathfrak{B}$ are abelian and therefore cyclic, of order dividing n. Now for each divisor d of n there is, up to linear isomorphism, just one irreducible linear group over F_p which is cyclic of order d, namely the group of dth roots of unity, operating as multipliers on the extension field of F_p they generate. If we denote this group by Z_d, it is clear that Z_d is a factor of $Z_{d'}$ if and only if d divides d'.

It follows that the closed classes of irreducible groups, and hence the varieties contained in $\mathfrak{B}$ and containing $\mathfrak{B}$, are in one-to-one correspondence with the trivially ordered subsets of the lattice of divisors of n. As defining identical relations for the variety we may take the relations of $\mathfrak{B}$ together with

$$(x^n, y^{d_1}, \ldots, y^{d_k}) = 1,$$

where $(d_1, \ldots, d_k)$ is the corresponding set of divisors of n, or $x^n = 1$ if this set is empty. It is then clear that $\mathfrak{B}$ is determined by its two generator groups, and in particular, therefore, is generated by them.

Example 4.10. Now let $\mathfrak{A} = {}_p\mathfrak{A}$ again, but let $\mathfrak{B}$ be the variety of groups with a central elementary abelian q-group whose factor group is again an elementary abelian q-group, where q is a prime different from p. Thus $\mathfrak{B}$ can be defined by the identical relations

$$x^{q^2p} = 1, \quad (x, y)^{qp} = 1, \quad (x, y, z)^p = 1, \quad (x^{q^2}, y^{q^2}) = 1.$$

We wish to determine the irreducible linear groups over F_p belonging to $\mathfrak{B}$. The argument is essentially a repetition of that contained in (**2**) § 2.4, and we omit it, giving only what we need of the conclusion. The abelian irreducible groups in $\mathfrak{B}$ are of course the cyclic groups of orders 1, q, q^2, which in this context we denote by P_0, P_1, P_2. Next come the non-abelian groups of order q^3, which we denote by P_3 and Q_3. If $q = 2$, it is immaterial which is which, but for odd q we insist that P_3 is the group containing elements of order q^2. For $r > 1$, we define inductively two groups, each of order q^{2r+1} with centre of order q, namely P_{2r+1}, which is the central product (direct product with identified centres) of P_{2r-1} and Q_3, and Q_{2r+1}, which is the central product of Q_{2r-1} and Q_3. We also define, for $r > 1$, P_{2r} to be the direct product of

P_{2r-1} and P_2, identifying the centre of P_{2r-1} with the subgroup of order q of P_2. Then, up to linear isomorphism, each group P_i or Q_i has just one representation as an irreducible linear group over F_p, and these exhaust the irreducible linear groups over F_p belonging to $\mathfrak{B}$. It is clear that, if $i < j$, then P_i is a factor of P_j, and Q_i of Q_j; and it is also true that Q_i is a factor of P_j. When, however, we consider whether P_i is a factor of Q_j, we find an essential difference between the cases $q = 2$ and q odd, owing to the fact that the central product of two copies of P_3 is in the one case isomorphic to Q_5 and in the other to P_5. In fact, if $q = 2$, P_i is a factor of Q_j whenever $i < j$, but for odd q only if $i = 0$ or 1. Thus in considering closed classes of irreducible groups, we must take the two cases separately.

If $q = 2$, and $\mathfrak{X}$ is a closed class of irreducible groups over F_p belonging to $\mathfrak{B}$, we denote by i the smallest integer such that P_i or Q_i fails to belong to $\mathfrak{X}$, or ∞ if there is no such integer. Then, unless i is an odd integer greater than 2, $\mathfrak{X}$ is determined by i, and we write it $\mathfrak{X}_i$. If i is an odd integer greater than 2, then there are three possibilities for $\mathfrak{X}$ since it may fail to contain P_i or Q_i or both. We may denote these by $\mathfrak{X}_i^{(P)}$, $\mathfrak{X}_i^{(Q)}$, and $\mathfrak{X}_i^{(0)}$ respectively. Evidently, of two closed classes with distinct suffixes, that with the smaller suffix is contained in the other; and $\mathfrak{X}_i^{(0)}$ is included in each of $\mathfrak{X}_i^{(P)}$ and $\mathfrak{X}_i^{(Q)}$, which are incomparable.

For odd q, a closed class $\mathfrak{X}$ is determined by i and j, which are respectively either ∞ or the least integers such that P_i or Q_j does not belong to $\mathfrak{X}$; and we write $\mathfrak{X}_{ij}$ for $\mathfrak{X}$. Here i is any integer or is ∞, j is an odd integer greater than 2 or is ∞, $i \leqslant j+1$, and $2 \leqslant i$ if $5 \leqslant j$. Evidently $\mathfrak{X}_{ij}$ is contained in $\mathfrak{X}_{ab}$ if $i \leqslant a$ and $j \leqslant b$.

In either case, by Theorem 4.6, the varieties contained in $\mathfrak{B}$ and containing $\mathfrak{B}$ are in one-to-one correspondence, with the closed classes, under a correspondence that preserves inclusion. I shall not attempt to write down the identical relations that distinguish between these varieties. This would not be uninteresting, but part of the point of the present treatment is to make it unnecessary.

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ON THE SUMMABILITY OF A POWER SERIES ON ITS CIRCLE OF CONVERGENCE

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1. This paper is concerned with a group of inequalities which are associated with the absolute summability and the strong summability of a power series on its circle of convergence. I have already discussed at some length the inequalities associated with absolute summability, but by considering these as part of the larger group of inequalities, we can throw some light on the nature of all these inequalities and their relations to each other. The paper is partly in the nature of a survey, but I have also attempted to fill some gaps which the survey discloses. Moreover, I give new and simplified proofs of several known results.

2. We begin with some notation. A function $\phi(z)$, regular in $|z| < 1$, will be said to be of 'class H^λ', where $\lambda > 0$, if

$$\left\{ \int_{-\pi}^{\pi} |\phi(\rho e^{i\theta})|^\lambda \, d\theta \right\}^{1/\lambda} \tag{2.1}$$

is bounded for all $\rho < 1$. It is familiar† that, if ϕ belongs to H^λ, then it possesses p.p. a non-tangential limit $\phi(e^{i\theta})$ as $z \to e^{i\theta}$, and $\phi(e^{i\theta})$ is of class L^λ in $(-\pi, \pi)$. Moreover, the expression (2.1) is a non-decreasing function of ρ and tends to the limit

$$\|\phi\|_\lambda = \left\{ \int_{-\pi}^{\pi} |\phi(e^{i\theta})|^\lambda \, d\theta \right\}^{1/\lambda}$$

as $\rho \to 1$.

We write also, for any f of class L^λ in $(-\pi, \pi)$,

$$\|f\|_\lambda = \left\{ \int_{-\pi}^{\pi} |f(\theta)|^\lambda \, d\theta \right\}^{1/\lambda}.$$

For any number k, used as an index (exponent) and such that $k > 1$, we write $k' = k/(k-1)$, so that k and k' are conjugate indices in the sense of Hölder's inequality. To avoid the necessity of separating

† See, for example, Zygmund (**25**) chapter 7.

various cases, we also adopt the convention that $1/k' = 0$ when $k = 1$ and that $1/k' = 1$ when $k = +\infty$.

We use $A(b, c, \ldots)$ to denote positive constants depending only on b, c, ...; A by itself will denote a positive absolute constant.

Inequalities of the form

$$L \leqslant A(b, c, \ldots)R$$

are to be interpreted as meaning 'if the expression R is finite, then the expression L is finite and satisfies the inequality'.

We also make frequent use of the Cesàro means of a series or sequence. Definitions and a list of identities can be found in (**8**), to which the reader is referred.

Let now $\phi(z)$ be regular in $|z| < 1$, let $\phi(z) = \sum_0^\infty c_n z^n$, and let $\sigma_n^\alpha(\theta)$ and $\tau_n^\alpha(\theta)$ be the nth (C, α) means of the series $\sum_0^\infty c_n e^{ni\theta}$ and the sequence $nc_n e^{ni\theta}$, respectively. The series $\sum c_n e^{ni\theta}$ is said to be 'absolutely summable (C, α) with index k', or 'summable $|C, \alpha|_k$', if the series

$$\sum n^{k-1} |\sigma_n^\alpha(\theta) - \sigma_{n-1}^\alpha(\theta)|^k = \sum n^{-1} |\tau_n^\alpha(\theta)|^k$$

converges, and to be 'absolutely summable (A) with index k', or 'summable $|A|_k$', if the integral

$$\int_0^1 (1-\rho)^{k-1} |\phi'(\rho e^{i\theta})|^k \, d\rho$$

is finite.† It is said to be 'strongly summable $(C, \alpha+1)$ with index k', or 'summable $\{\mathbf{C}, \alpha\}_k$', to the sum $S(\theta)$ if

$$\sum_0^m |\sigma_n^\alpha(\theta) - S(\theta)|^k = o(m)$$

as $m \to \infty$, and to be 'strongly summable (A) with index k', or 'summable $\{\mathbf{A}\}_k$', to $S(\theta)$ if

$$\int_0^R \frac{|\phi(\rho e^{i\theta}) - S(\theta)|^k}{(1-\rho)^2} \, d\rho = o\left(\frac{1}{1-R}\right)$$

as $R \to 1-$.‡ If

$$\sum_0^m |\sigma_n^\alpha(\theta)|^k = O(m) \quad \text{or} \quad \int_0^R \frac{|\phi(\rho e^{i\theta})|^k}{(1-\rho)^2} \, d\rho = O\left(\frac{1}{1-R}\right),$$

† For the general theory of summabilities $|C, \alpha|_k$ and $|A|_k$ see (**4**, **5**, **6**, **7**).

‡ For the general theory of summabilities $\{\mathbf{C}, \alpha\}_k$, $\{\mathbf{A}\}_k$, $\{C, \alpha\}_k$, and $\{A\}_k$ see (**8**).

we say also that $\sum c_n e^{ni\theta}$ is 'bounded $\{\mathbf{C}, \alpha\}_k$' and 'bounded $\{\mathbf{A}\}_k$', respectively. Finally, we say that $\sum c_n e^{ni\theta}$ is 'summable $\{C, \alpha\}_k$' or 'bounded $\{C, \alpha\}_k$' according as

$$\sum_1^m |\tau_n^\alpha(\theta)|^k = o(m) \quad \text{or} \quad O(m),$$

and that it is 'summable $\{A\}_k$' or 'bounded $\{A\}_k$' according as

$$\int_0^R (1-\rho)^{k-2} |\phi'(\rho e^{i\theta})|^k \, d\rho = o\left(\frac{1}{1-R}\right) \quad \text{or} \quad O\left(\frac{1}{1-R}\right).$$

In all these we suppose that $k \geqslant 1$ and $\alpha > -1$.

With each form of summability defined above we can associate a function containing σ_n^α, τ_n^α, or ϕ. Thus with summability $|C, \alpha|_k$ and summability $|A|_k$ we associate the functions

$$h_{k,\alpha}(\theta) = \left\{ \sum_1^\infty n^{-1} |\tau_n^\alpha(\theta)|^k \right\}^{1/k},$$

and

$$g_k(\theta) = \left\{ \int_0^1 (1-\rho)^{k-1} |\phi'(\rho e^{i\theta})|^k \, d\rho \right\}^{1/k},$$

respectively. With summabilities $\{\mathbf{C}, \alpha\}_k$ and $\{\mathbf{A}\}_k$ (more precisely, with boundedness $\{\mathbf{C}, \alpha\}_k$ and $\{\mathbf{A}\}_k$) we associate the functions

$$\mathbf{H}_{k,\alpha}(\theta) = \sup_m \left\{ \frac{1}{m+1} \sum_0^m |\sigma_n^\alpha(\theta)|^k \right\}^{1/k},$$

and

$$\mathbf{G}_k(\theta) = \sup_{0<R<1} \left\{ (1-R) \int_0^R (1-\rho)^{-2} |\phi(\rho e^{i\theta})|^k \, d\rho \right\}^{1/k},$$

respectively, while with summabilities $\{C, \alpha\}_k$ and $\{A\}_k$ (or, rather, boundedness) we associate the functions

$$H_{k,\alpha}(\theta) = \sup_m \left\{ \frac{1}{m+1} \sum_1^m |\tau_n^\alpha(\theta)|^k \right\}^{1/k}$$

and

$$G_k(\theta) = \sup_{0<R<1} \left\{ (1-R) \int_0^R (1-\rho)^{k-2} |\phi'(\rho e^{i\theta})|^k \, d\rho \right\}^{1/k}.$$

Here, as before, $k \geqslant 1$ and $\alpha > -1$. We have also four functions which are the limiting forms of these expressions as $k \to \infty$, namely

$$h_{\infty,\alpha}(\theta) = H_{\infty,\alpha}(\theta) = \sup_n |\tau_n^\alpha(\theta)|,$$

$$g_\infty(\theta) = G_\infty(\theta) = \sup_\rho \{(1-\rho)|\phi'(\rho e^{i\theta})|\},$$

and

$$\mathbf{H}_{\infty,\alpha}(\theta) = \sup_n |\sigma_n^\alpha(\theta)|, \qquad \mathbf{G}_\infty(\theta) = \sup_\rho |\phi(\rho e^{i\theta})|.$$

We shall refer to h, H, and $\mathbf{H}$ as *Cesàro-type functions*, and to g, G, and $\mathbf{G}$ as *Abel-type functions*.

To these functions we add two more, namely

$$\Phi(\theta) = \sup|\phi(z)| \quad (z \in \Omega(\theta))$$

and

$$s(\theta) = \left\{ \iint_{\Omega(\theta)} |\phi'(z)|^2 \, d\omega \right\}^{\frac{1}{2}},$$

where $d\omega$ is the element of area and $\Omega(\theta)$ is the kite-shaped region in $|z| < 1$ bounded by the tangents drawn to the circle $|z| = \sin\eta < 1$ from the point $z = e^{i\theta}$ together with the further arc of this circle.† The expression $s^2(\theta)$ is, of course, the area of the image of Ω under the mapping $w = \phi(z)$, multiple points being counted according to their multiplicity, while $\Phi(\theta)$ is the radius of the smallest circle with centre at $w = 0$ containing this image.

3. There are a number of inequalities connecting these functions, valid for each θ, which are immediate consequences of various Abelian theorems proved in (**4**) and (**8**). Thus, for $1 \leqslant k \leqslant \infty$ and $\alpha > -1$, we have

$$G_k \leqslant g_k, \qquad H_{k,\alpha} \leqslant h_{k,\alpha}, \tag{3.1}$$

$$H_{k,\alpha+1} \leqslant A(k,\alpha)\mathbf{H}_{k,\alpha}. \tag{3.2}$$

If $1 \leqslant k \leqslant l \leqslant \infty$ and $\alpha > -1$, then

$$G_k \leqslant G_l, \quad H_{k,\alpha} \leqslant H_{l,\alpha}, \quad \mathbf{G}_k \leqslant \mathbf{G}_l, \quad \mathbf{H}_{k,\alpha} \leqslant \mathbf{H}_{l,\alpha}. \tag{3.3}$$

Again, if $\mathscr{H}$ denotes any one of the three Cesàro-type functions, we have

$$\mathscr{H}_{r,\beta} \leqslant A(k,r,\alpha,\beta)\mathscr{H}_{k,\alpha} \tag{3.4}$$

for all $\alpha > -1$, provided that either

(i) $1 \leqslant k \leqslant r \leqslant \infty$ and $\beta > \alpha + 1/k - 1/r$,

(ii) $1 < k < r < \infty$ and $\beta = \alpha + 1/k - 1/r$,

or (iii) $k = 1$, $r = \infty$ and $\beta \geqslant \alpha + 1$.

Further, if $\mathscr{G}$ and $\mathscr{H}$ are any two associated Abel-type and Cesàro-type functions, we have

$$\mathscr{G}_k \leqslant A(k,\alpha)\mathscr{H}_{k,\alpha} \tag{3.5}$$

for $1 \leqslant k \leqslant \infty$ and $\alpha > -1$.

We have also relations connecting the Ω-functions with the Cesàro-

† More precisely, $\Omega(\theta)$ is defined by the relations

$$|z| < 1, \quad |z| \leqslant \sin\eta \quad \text{or} \quad |z - e^{i\theta}| \leqslant \cos\eta, \quad |\arg(e^{i\theta} - z) - \theta| \leqslant \eta,$$

where $0 < \eta < \frac{1}{2}\pi$.

type and Abel-type functions, namely

$$g_k \leqslant A(k,\eta)s \quad (2 \leqslant k \leqslant \infty), \tag{3.6}$$

$$s \leqslant A(\alpha,\eta)h_{2,\alpha} \quad (\alpha > -1), \tag{3.7}$$

$$g_\infty = G_\infty \leqslant A(\eta)\Phi, \qquad \mathbf{G}_\infty \leqslant \Phi, \tag{3.8}$$

and

$$\Phi \leqslant A(\eta)\mathbf{H}_{k,\alpha} \quad (1 \leqslant k \leqslant \infty,\ \alpha > -1). \tag{3.9}$$

Here the relations (3.1) are immediate, (3.2) is the O-form of Theorem 6 of (**8**), and (3.3) follows from Theorem 1 of (**8**). The case $k = r = \infty$ of (3.4) is trivial, and the remaining cases when $\mathscr{H}$ is either H or $\mathbf{H}$ are consequences of Theorems 2 and 3 of (**8**). The case $r < \infty$ for $\mathscr{H} = h$ is proved in (**4**), and the case $k < r = \infty$ for $\mathscr{H} = h$ is a consequence of the same case for $\mathscr{H} = H$ and of (3.1). The cases $k = \infty$ of (3.5) are immediate, the case $k < \infty$ when $\mathscr{G}$ is either G or $\mathbf{G}$ is a consequence of Theorems 1 and 3 of (**8**), and the case $k < \infty$ when $\mathscr{G} = g$ is proved in (**4**). Finally, (3.6)–(3.9) are proved in § 15 of (**8**).

There are also inequalities of a Tauberian character, again valid for each θ; in particular we have Tauberian results of the form

$$\mathscr{H}_{k,\alpha} \leqslant A(k)\mathscr{G}_k + A(k,\alpha)\mathscr{J}_{k,\alpha} \tag{3.10}$$

corresponding to the Abelian result (3.5). Here $\mathscr{G}$ and $\mathscr{H}$ are any associated Abel-type and Cesàro-type functions, and

$$\mathscr{J}_{k,\alpha} = \left\{\sum_1^\infty n^{-1}|\mu_n^{\alpha+1}|^k\right\}^{1/k}, \quad \sup_m\left\{\frac{1}{m+1}\sum_1^m |\mu_n^{\alpha+1}|^k\right\}^{1/k}, \quad \text{or } H_{k,\alpha+1}$$

according as $\mathscr{G}$ is g, G, or $\mathbf{G}$, where $\mu_n^{\alpha+1}$ is the nth $(C,\alpha+1)$ mean of the sequence†

$$n\{nc_n e^{ni\theta} - (n-1)c_{n-1}e^{(n-1)i\theta}\}.$$

4. We have also a number of integral inequalities, which fall into two classes. First there are inequalities of the form

$$\|\mathscr{H}_{k,\alpha}\|_\lambda \leqslant A(k,\alpha,\lambda)\|\mathscr{G}_k\|_\lambda$$

connecting a Cesàro-type function $\mathscr{H}$ with the appropriate Abel-type function $\mathscr{G}$, valid for certain k, α, and λ. An inequality of this type, which I shall call a '$\mathscr{G}\mathscr{H}$-inequality', is essentially an 'averaged' Abelian theorem, and asserts that on the average $\mathscr{H}_{k,\alpha}$ does not exceed some multiple of $\mathscr{G}_k$, a result which is in general false for any particular θ [cf. (3.5) and (3.10)].

These $\mathscr{G}\mathscr{H}$-inequalities are decidedly non-trivial, and only partial results are known. Thus we have

$$\|h_{k,\alpha}\|_\lambda \leqslant A(k,\alpha,\lambda)\|g_k\|_\lambda \tag{4.1}$$

† (**7**) Theorems 3 and 5.

for $1 < k < \infty, \lambda > 1$, and $\alpha \geqslant 1$† or for $\lambda \geqslant k > 1$ and $\alpha > \sup(1/k, 1/k')$.‡ We have also

$$\|\mathbf{H}_{\infty,\alpha}\|_\lambda \leqslant A(\alpha,\lambda)\|\mathbf{G}_\infty\|_\lambda \tag{4.2}$$

for $\lambda > 0$ and $\alpha > \sup(0, 1/\lambda - 1)$.§

We are able to complete the **GH**-inequalities: we prove in fact that

$$\|\mathbf{H}_{k,\alpha}\|_\lambda \leqslant A(k,\alpha,\lambda)\|\mathbf{G}_k\|_\lambda \tag{4.3}$$

for $1 \leqslant k \leqslant \infty$, $\lambda > 0$, and $\alpha > -1 + \sup(1/k', 1/\lambda, \frac{1}{2})$.

Our knowledge of the *GH*-inequalities, like those of the *gh*-inequalities, is less complete, and here we can prove only that

$$\|H_{k,\alpha}\|_\lambda \leqslant A(k,\alpha,\lambda)\|G_k\|_\lambda \tag{4.4}$$

for either $k = \infty, \lambda > 1$, and $\alpha > 1$ or $\lambda > k \geqslant 1$ and $\alpha > \sup(1/k', 1/\lambda, \frac{1}{2})$.‖

The existence of (4.3) and (4.4) shows that (4.1) is not just an isolated curiosity, but one of a general class of results associated with these various methods of summability. We can also make the 'average' aspect more explicit in the case of these inequalities. Thus for instance we can show that, for $k \geqslant 1$ and $\alpha > \sup(1/k, 1/k')$,

$$H_{k,\alpha}^k(\theta) \leqslant A(k,\alpha) \sup_{0<|u|\leqslant\pi} \left\{ \frac{1}{u} \int_0^u G_k^k(\theta+v)\, dv \right\}. \tag{4.5}$$

A similar inequality holds between **H** and **G**.

5. The second class of integral inequality consists of inequalities connecting the various expressions $\mathscr{G}$ and $\mathscr{H}$ with $\phi(e^{i\theta})$, and here our information is much more complete. First we have inequalities relating the $\mathscr{G}$'s with ϕ, which I shall call '$\mathscr{G}$-inequalities'. These are

$$\|g_k\|_\lambda \leqslant A(k,\lambda)\|\phi\|_\lambda \quad (2 \leqslant k \leqslant \infty;\ \lambda > 0), \tag{5.1}$$

$$\|G_k\|_\lambda \leqslant A(k,\lambda)\|\phi\|_\lambda \quad (1 \leqslant k \leqslant \infty;\ \lambda > 0), \tag{5.2}$$

$$\|\mathbf{G}_k\|_\lambda \leqslant A(k,\lambda)\|\phi\|_\lambda \quad (1 \leqslant k \leqslant \infty;\ \lambda > 0), \tag{5.3}$$

$$\|\phi\|_\lambda \leqslant A(k,\lambda)\|g_k\|_\lambda \quad (1 \leqslant k \leqslant 2;\ \lambda > 1;\ \phi(0) = 0). \tag{5.4}$$

† This case is contained implicitly in work of Zygmund and Sunouchi. For references see (**4**).

‡ (**4**) Theorem 12.

§ By the Hardy–Littlewood 'Complex Max' theorem [see (5.5)], we may replace $\mathbf{G}_\infty$ on the right of (4.2) by $\phi(e^{i\theta})$, and in this form the inequality is familiar [see (**4**) and references given there].

‖ In (4.4), and also in (4.1), we are unable to use the familiar reduction to the case of a zero-free ϕ, and thence, by the transformation $\psi(z) = \{\phi(z)\}^\mu$, to a case already proved. This is why we cannot extend the results to all $\lambda > 0$.

To these we may add the Ω-results

$$\|\Phi\|_\lambda \leqslant A(\lambda)\|\phi\|_\lambda \quad (\lambda > 0), \tag{5.5}$$

$$\|s\|_\lambda \leqslant A(\lambda)\|\phi\|_\lambda \quad (\lambda > 0). \tag{5.6}$$

Here (5.1) and (5.4) are the Littlewood–Paley 'g' inequalities,† (5.5) is the Hardy–Littlewood 'Complex Max' theorem (**9**), and (5.6) is an inequality due to Marcinkiewicz and Zygmund (**17**). The inequalities (5.2) and (5.3) are immediate consequences of (5.5), (3.8), and (3.3).‡ We may also deduce (5.2) from (5.1), (3.1), and (3.3). We note also that (5.1) is an immediate consequence of (5.6) and (3.6).§

The inequalities (5.1), (5.2), and (5.3) are the most natural means of proving that, if ϕ belongs to H^λ, then for almost all θ the series $\sum c_n e^{ni\theta}$ is summable $|A|_k$, bounded $\{A\}_k$, and bounded $\{\mathbf{A}\}_k$, respectively. To prove such a result we have to prove that the appropriate Abel-type function is finite p.p., and to do this it is natural to attempt to prove that this function belongs to some Lebesgue class L^λ. The actual form of the inequality is then suggested by considerations of homogeneity. There is, however, an alternative point of view. In virtue of the trivial inequality

$$|\phi(e^{i\theta})| \leqslant \mathbf{G}_\infty(\theta) \tag{5.7}$$

and the case $k = \infty$ of (5.3), we may replace ϕ throughout (5.1)–(5.6) by $\mathbf{G}_\infty$, and then these inequalities fall into the same category as the $\mathscr{GH}$-inequalities of § 4, i.e. they are essentially 'averaged' Abelian theorems connecting the various Abel-type properties (or Ω-properties) with boundedness (A).‖

We have one new inequality to add here, namely

$$\|\phi\|_\lambda \leqslant A(k,\lambda)\|\mathbf{G}_k\|_\lambda \quad (1 \leqslant k < \infty;\ \lambda > 0). \tag{5.8}$$

This, together with (5.3) and (5.7), completes the set of G-inequalities.

We observe also that the corresponding G-inequality, namely

$$\|\phi\|_\lambda \leqslant A(k,\lambda)\|G_k\|_\lambda,$$

† Littlewood and Paley (**15**). For other references see (**4**).

‡ This proof of (5.2) (for $k = \infty$ and so for all k) is due to Littlewood and Paley (**15**) 58.

§ It does not appear to have been observed hitherto that (5.6) implies (5.1) for $k > 2$. For a proof of (5.6) see Flett (**3**).

‖ We recall here a result of Spencer's (**20**) that, if $s(\theta)$ is finite in a set E of θ of positive measure (the angle η of $\Omega(\theta)$ varying with θ), then ϕ possesses a non-tangential limit p.p. in E.

is false for every $k \geqslant 1$ and every $\lambda > 0$. For, if

$$\phi(z) = \sum_0^\infty z^{a^n},$$

where a is any integer exceeding e^3, then†

$$|\phi'(\rho e^{i\theta})| \leqslant A(a)(1-\rho)^{-1} \tag{5.9}$$

for $\rho < 1$ and all θ, so that $G_\infty(\theta)$ is bounded (whence also G_k is bounded for every k). On the other hand, there exists a sequence $\rho_n \to 1$ such that

$$|\phi'(\rho_n e^{i\theta})| \geqslant A(a)(1-\rho_n)^{-1} \tag{5.10}$$

for all n and θ. Hence also

$$\left\{ \int_{-\pi}^{\pi} |\phi'(\rho_n e^{i\theta})|^\lambda \, d\theta \right\}^{1/\lambda} \geqslant A(a)(1-\rho_n)^{-1}$$

for any $\lambda > 0$, and so, by a theorem due to Hardy and Littlewood [(**10**) Theorem 46], ϕ does not belong to H^λ for any $\lambda > 0$.

We note finally that there are two outstanding questions here, namely whether the inequality (5.4) continues to hold either for $1 < k \leqslant 2$ and $0 < \lambda \leqslant 1$‡ or for $k > 2$ and any $\lambda > 0$.

This second class of integral inequalities contains also inequalities, which I shall call '$\mathscr{H}$-inequalities', of the type§

$$\|\mathscr{H}_{k,\alpha}\|_\lambda \leqslant A(k,\alpha,\lambda)\|\phi\|_\lambda.$$

We have in fact‖

$$\|h_{k,\alpha}\|_\lambda \leqslant A(k,\alpha,\lambda)\|\phi\|_\lambda \quad (2 \leqslant k \leqslant \infty;\ \lambda > 0;\ \alpha > \sup(1/k', 1/\lambda)), \tag{5.11}$$

$$\|H_{k,\alpha}\|_\lambda \leqslant A(k,\alpha,\lambda)\|\phi\|_\lambda \quad (1 \leqslant k \leqslant \infty;\ \lambda > 0;\ \alpha > \sup(1/k',\ 1/\lambda, \tfrac{1}{2})), \tag{5.12}$$

$$\|\mathbf{H}_{k,\alpha}\|_\lambda \leqslant A(k,\alpha,\lambda)\|\phi\|_\lambda \quad (1 \leqslant k \leqslant \infty;\ \lambda > 0;\ \alpha > -1+\sup(1/k', 1/\lambda, \tfrac{1}{2})). \tag{5.13}$$

These $\mathscr{H}$-inequalities are, of course, stronger than the $\mathscr{G}$-inequalities (5.1)–(5.3), but (except for $k = \infty$) their proof actually proceeds via the $\mathscr{G}$-inequalities. They should presumably be immediate corollaries

† For a proof of (5.9) and (5.10) see Littlewood (**14**) § 8.5.

‡ For a partial result in this direction see (**3**).

§ The reversed inequalities (when they exist) are immediate consequences of the corresponding $\mathscr{G}$-inequalities [(5.4) and (5.8)] and (3.5), and require no further mention.

‖ In view of the similarities between (5.1) and (5.3), (5.4) and (5.8), and (5.11) and (5.13), it is tempting to look for simple inequalities of the type listed in § 3 connecting $\mathbf{G}_k$ with g_k and $\mathbf{H}_{k,\alpha}$ with $h_{k,\alpha+1}$. However, there do not appear to be any such inequalities [see (**8**) § 14].

of the $\mathscr{G}$- and $\mathscr{GH}$-inequalities, and this is in fact so for $\lambda > k$. In the case of (5.11) and (5.12), the absence of complete information about the $\mathscr{GH}$-inequalities prevents our using this approach for all $\lambda > 0$. In the case of (5.13), the corresponding $\mathscr{GH}$-inequality (4.3) does hold for $\lambda > 0$, but we cannot deduce (5.13) from (4.3) when $0 < \lambda \leqslant k$, for we actually use (5.13) in the proof of (4.3) for these values of λ.

The case $k < \infty$ of (5.11) is proved in (**4**), while the case $k = \infty$, which is also the case $k = \infty$ of (5.12), is proved in (**6**).† The case $k \geqslant 2$ of (5.12) is an immediate consequence of (5.11) and (3.1), and the case $1 \leqslant k < 2$ follows easily from the case $k \geqslant 2$. Further, (5.13) is a consequence of (5.3), (5.12), and the case $\mathscr{G} = \mathbf{G}$ of (3.10). The inequalities (5.12) and (5.13) can also be proved independently of each other and of (5.11), and these independent proofs are a good deal easier than is the proof of the case $k < \infty$ of (5.11).‡

We note that, in virtue of (5.8), the result of (4.3) is a consequence of (5.13).§

I have a little to add here concerning these $\mathscr{H}$-inequalities. Thus I give here a new and simplified proof of the case $\lambda \leqslant k$ of (5.11). We note also that the arguments of § 7 give a slightly improved proof of the case $k \geqslant 2$ of (5.12) and (5.13). Further, if we combine either (5.12) or (5.13) with Theorem 11 of (**8**), we obtain the following result.

THEOREM 1. *If* $\phi(z) = \sum c_n z^n$ *belongs to* H^λ *in* $|z| < 1$, *where* $\lambda > 0$, *and, if* $1 \leqslant k \leqslant \infty$ *and* $\alpha > -1 + \sup(1/k', 1/\lambda, \frac{1}{2})$, *then* $\sum c_n e^{ni\theta}$ *is summable* $\{\mathbf{C}, \alpha\}_k$ *to* $\phi(e^{i\theta})$ *p.p.*

The case $k = \infty$ of this theorem is well known. The case $\lambda > 1$, $k < \infty$ is due to Chow (**2**), while the general case $\lambda > 0$, $k < \infty$ was proved by the author in (**6**). The argument given in (**6**), however, makes use of the inequality (5.11) instead of the simpler (5.12) and (5.13).

6. We pass finally to the limiting cases of the $\mathscr{GH}$- and $\mathscr{H}$-inequalities, corresponding to the limiting values of α in (4.1)–(4.4) and (5.11)–(5.13). Here we have a number of different types of results according to whether we weaken the conclusion or strengthen the hypothesis. As an example

† For various cases of these inequalities see references given in (**4**), and also Koizumi (**12**, **13**), Sunouchi (**21**, **22**), Sunouchi and Yano (**23**), and Zygmund (**28**).

‡ A proof of the case $k < \infty$ of (5.12) independent of (5.11) is given in (**6**), and an independent proof of (5.13) can be constructed on similar lines to this.

§ Our arguments, however, proceed partly in the opposite direction.

of the first type we have

$$\left\|\sup_{n\geqslant 1}\{|\sigma_n^{\alpha}(\theta)|/\log^{\delta}(n+1)\}\right\|_{\lambda}\leqslant A(\lambda)\|\phi\|_{\lambda} \tag{6.1}$$

if (i) $0<\lambda\leqslant 1$, $\alpha=1/\lambda-1$, $\delta=1/\lambda$, (ii) $1<\lambda\leqslant 2$, $\alpha=0$, $\delta=1/\lambda$, or (iii) $2<\lambda\leqslant\infty$,† $\alpha=0$, $\delta=1/\lambda'$. Further,

$$\left\|\Big\{\sum_{1}^{\infty} n^{-1}(\log(n+1))^{-2\lambda}|\tau_n^{1/\lambda}(\theta)|^2\Big\}^{\frac{1}{2}}\right\|_{\lambda}\leqslant A(\lambda)\|\phi\|_{\lambda} \tag{6.2}$$

if $0<\lambda\leqslant 2$. Also

$$\left\|\sup_{n}|\sigma_n^{1/\lambda-1}(\theta)|\right\|_{\lambda\mu}\leqslant A(\lambda,\mu)\|\phi\|_{\lambda} \tag{6.3}$$

for $0<\lambda\leqslant\frac{1}{2}$ and $0<\mu<1$, and

$$\|h_{2,1/\lambda}(\theta)\|_{\lambda\mu}\leqslant A(\lambda,\mu)\|\phi\|_{\lambda} \tag{6.4}$$

for $0<\lambda\leqslant 1$ and $0<\mu<1$.

As examples of the second type we have

$$\int_{-\pi}^{\pi}\sup_{n}|\sigma_n^{1/\lambda-1}(\theta)|^{\lambda}\,d\theta\leqslant A(\lambda)\int_{-\pi}^{\pi}|\phi(e^{i\theta})|^{\lambda}\log^{+}|\phi(e^{i\theta})|\,d\theta+A(\lambda) \tag{6.5}$$

for $0<\lambda\leqslant\frac{1}{2}$, and

$$\int_{-\pi}^{\pi}h_{2,1/\lambda}^{\lambda}(\theta)\,d\theta\leqslant A(\lambda)\int_{-\pi}^{\pi}|\phi(e^{i\theta})|^{\lambda}\log^{+}|\phi(e^{i\theta})|\,d\theta+A(\lambda) \tag{6.6}$$

for $0<\lambda\leqslant 1$.

Here cases (ii) and (iii) of (6.1) are due to Littlewood and Paley (**16**), the case $\lambda=1$ of (i) to Zygmund (**26**) and the rest of case (i) to Sunouchi (**21**, **22**). The cases $\lambda=1$ of (6.4) and (6.6) were proved by Zygmund in (**26**), while the cases $0<\lambda<1$ of these, and also the inequalities (6.3) and (6.5), were proved independently by Sunouchi (**21**) and Zygmund (**28**). Finally the case $\lambda=2$ of (6.2) is due to Zygmund (**26**) and the case $0<\lambda<2$ to Sunouchi (**21**, **22**).

Of all these inequalities, the most difficult seems to be (6.1). The case $2<\lambda<\infty$ is proved by an application of M. Riesz's interpolation theorem between $\lambda=2$ and $\lambda=\infty$. The case $1<\lambda<2$ can also be proved by interpolation between $\lambda=1$ and $\lambda=2$, by using a form of interpolation suitable for the classes H^{λ}.‡ The critical cases are therefore $0<\lambda\leqslant 1$, $\lambda=2$, and $\lambda=\infty$. The last of these is trivial, while for the second there are available relatively short (though not simple)

† This is the only point at which we consider $\lambda=\infty$.

‡ Salem and Zygmund (**19**), Calderón and Zygmund (**1**). The original proof of Littlewood and Paley for the case $1<\lambda<2$ is extremely complicated.

arguments.† On the other hand, Sunouchi's proof of the case $0 < \lambda \leqslant 1$ of (6.1), which proceeds via (6.2), is very complicated. It seems therefore of interest to give a new proof of (6.1) for $0 < \lambda \leqslant 1$, and this I do in § 10. It will be seen that in this proof we make no use of the oscillating factor in the integrand in the integral representation for $\sigma_n^\alpha(\theta)$. The proof of the case $\lambda = 2$ of (6.1), however, seems to require some consideration of this oscillating factor, and is therefore considerably more delicate.

At the same time I give a new proof of (6.2) and complete this result for all $k \geqslant 2$. We prove in fact that

$$\|\{\sum n^{-1}(\log(n+1))^{-k\delta}|\tau_n^\alpha(\theta)|^k\}^{1/k}\|_\lambda \leqslant A(k,\lambda)\|\phi\|_\lambda \tag{6.7}$$

for $k \geqslant 2, \lambda > 0, \alpha = \sup(1/k', 1/\lambda)$, and $\delta = \sup(1/k, 1/\lambda)$. From this we can easily deduce limiting forms of (5.12) and (5.13). For $\lambda \geqslant k$ and for appropriate α and δ we could also replace ϕ on the right of (6.7) by g_k, thus obtaining a limiting form of (4.1), and this, moreover, for $k > 1$.

Our arguments give also new proofs of (6.3) and (6.5) (the cases $\lambda = 1$ of (6.4) and (6.6) being taken as known), but since these present no fresh points of interest I omit them.

7. We pass now to the proof of the new results stated in §§ 4–5. For convenience we restate these as theorems, retaining the notation of § 2. We deal first with (4.4), which we restate as

THEOREM 2. *If either* $k = \infty$, $\lambda > 1$ *and* $\alpha > 1$ *or* $\lambda > k \geqslant 1$ *and* $\alpha > \sup(1/k', 1/\lambda, \frac{1}{2})$, *then*

$$\|H_{k,\alpha}\|_\lambda \leqslant A(k,\alpha,\lambda)\|G_k\|_\lambda.$$

The first case of this theorem follows by an obvious modification of the argument of Theorem 3 of (6), and we omit it. To prove the remaining case we require a number of lemmas. Some of these state more than is required in the proof of Theorem 2, but we require the complete versions later.

LEMMA 1. *Let* $\gamma \geqslant 0$, $\frac{1}{2} \leqslant R < 1$ *and* $\gamma(1-R) < \delta$. *Then, for all real* β,

$$\int_0^R (1-\rho)^{\beta-1}\rho^\gamma \, d\rho \geqslant A(\beta,\delta)(\gamma+1)^{-\beta}.$$

This is an improved version of Lemma 1 of (6).

If $\gamma \leqslant \delta+1$,

$$\int_0^R (1-\rho)^{\beta-1}\rho^\gamma \, d\rho \geqslant \int_0^{\frac{1}{2}} (1-\rho)^{\beta-1}\rho^{\delta+1} \, d\rho = A(\beta,\delta) \geqslant A(\beta,\delta)(\gamma+1)^{-\beta},$$

† Hardy and Littlewood (11), Yano (24).

the required result. Suppose then that $\gamma > \delta+1$. Writing $t = \gamma(1-\rho)$, we get

$$\int_0^R (1-\rho)^{\beta-1}\rho^\gamma \, d\rho = \gamma^{-\beta} \int_{\gamma(1-R)}^{\gamma} t^{\beta-1}\left(1-\frac{t}{\gamma}\right)^\gamma dt \geqslant \gamma^{-\beta} \int_\delta^{\delta+1} t^{\beta-1}\left(1-\frac{t}{\gamma}\right)^\gamma dt.$$

Since $(1-t/\gamma)^\gamma$ is an increasing function of γ for $\gamma > t$, and since $\gamma > \delta+1 \geqslant t$ in the last integral above, we have now

$$\int_0^R (1-\rho)^{\beta-1}\rho^\gamma \, d\rho \geqslant \gamma^{-\beta} \int_\delta^{\delta+1} t^{\beta-1}\left(1-\frac{t}{\delta+1}\right)^{\delta+1} dt$$

$$= A(\beta,\delta)\gamma^{-\beta} \geqslant A(\beta,\delta)(\gamma+1)^{-\beta},$$

again the required result.

LEMMA 2. *Let $\xi(\theta)$ be periodic with period 2π and integrable in $(-\pi,\pi)$, and let*

$$\xi^*(\theta) = \sup_{0<|u|\leqslant\pi} \left\{\frac{1}{u}\int_0^u |\xi(\theta+v)|\, dv\right\}. \tag{7.1}$$

Then, if $\lambda > 1$, $\|\xi^*\|_\lambda \leqslant \lambda'\|\xi\|_\lambda$.

This is a well-known result of Hardy and Littlewood (9).

LEMMA 3. *Let $\xi(\theta)$ be periodic with period 2π and integrable in $(-\pi,\pi)$, let $\xi^*(\theta)$ be defined by* (7.1), *and let*

$$I_\lambda(\rho,\theta) = \int_{-\pi}^{\pi} \frac{\xi(\theta+t)}{|1-\rho e^{it}|^\lambda}\, dt.$$

If $\lambda > 1$, then $|I_\lambda(\rho,\theta+\psi)| \leqslant \dfrac{A(\lambda)\xi^*(\theta)|1-\rho e^{i\psi}|}{(1-\rho)^\lambda}$.

If $\lambda = 1$, then $|I_\lambda(\rho,\theta)| \leqslant A\xi^*(\theta)\log\dfrac{2}{1-\rho}$.

The first result is Lemma 4 of (4). The proof of the second is similar to that of the first, and we omit it.

LEMMA 4. *Let $f(z)$ be regular and $f(z) = \sum_0^\infty d_n z^n$ in $|z| < 1$, let*

$$0 < l \leqslant k < \infty, \quad \beta = \frac{1}{k}+\frac{1}{l}+\gamma-1 \geqslant 0, \quad \text{and} \quad \gamma \geqslant 0.$$

Then

$$\left\{\sum_0^\infty (n+1)^{-k\beta}|d_n|^k\rho^{nk}\right\}^{1/k} \leqslant A(k,l,\gamma)\left\{\int_{-\pi}^{\pi} |f(\rho e^{it})|^l|1-\rho e^{it}|^{l\gamma}\, dt\right\}^{1/l}.$$

This is a well-known theorem of Pitt's (18). Pitt actually states the theorem for the case in which $d_0 = 0$, but this restriction is unnecessary.

The remaining two lemmas are concerned with a majorant of $H_{k,\alpha}$. For any σ and any R such that $0 < R < 1$ let

$$W_k(\sigma;\, R, \theta) = \left\{(1-R)\int_{-\pi}^{\pi}\int_{0}^{R}(1-\rho)^{k+\sigma-3}\frac{|\phi'(\rho e^{i\theta+it})|^k}{|1-\rho e^{it}|^{\sigma}}\,d\rho dt\right\}^{1/k}. \tag{7.2}$$

Then we have

LEMMA 5. *Let $1 \leqslant k < \infty$ and $\alpha > -1$, let $\gamma = 0$ or $1-2/k$ according as $k < 2$ or $k \geqslant 2$, and let $R = 1-1/(m+1)$. Then*†

$$\left\{\frac{1}{m+1}\sum_{1}^{m}|\tau_n^{\alpha}(\theta)|^k\right\}^{1/k} \leqslant A(k,\alpha)W_k(k\alpha-k\gamma;\, R, \theta). \tag{7.3}$$

By Lemma 4, we have

$$\sum_{0}^{\infty}(n+1)^{-k\beta}|d_n|^k\rho^{nk} \leqslant A(k)\int_{-\pi}^{\pi}|f(\rho e^{it})|^k|1-\rho e^{it}|^{k\gamma}\,dt$$

for any function $f(z) = \sum d_n z^n$ regular in $|z| < 1$, where either

$$1 \leqslant k < 2, \quad \beta = 2/k-1, \quad \gamma = 0,$$

or

$$2 \leqslant k < \infty, \quad \beta = 0, \quad \gamma = 1-2/k.$$

Applying this inequality to the function

$$\frac{z\phi'(ze^{i\theta})}{(1-z)^{\alpha}} = \sum_{1}^{\infty}E_n^{\alpha}\tau_n^{\alpha}(\theta)z^n,$$

we get

$$\sum_{1}^{m}(n+1)^{k\alpha-k\beta}|\tau_n^{\alpha}|^k\rho^{nk} \leqslant \sum_{1}^{\infty} \leqslant A(k,\alpha)\int_{-\pi}^{\pi}\frac{|\phi'(\rho e^{i\theta+it})|^k}{|1-\rho e^{it}|^{k\alpha-k\gamma}}\,dt. \tag{7.4}$$

Let $R = 1-1/(m+1)$. Then multiplying both sides of (7.4) by $(1-\rho)^{k\alpha-k\beta-1}$, integrating with respect to ρ from 0 to R, and using Lemma 1, we get

$$\frac{1}{m+1}\sum_{1}^{m}|\tau_n^{\alpha}|^k \leqslant A(k,\alpha)(1-R)\int_{0}^{R}\int_{-\pi}^{\pi}(1-\rho)^{k\alpha-k\beta-1}\frac{|\phi'(\rho e^{i\theta+it})|^k}{|1-\rho e^{it}|^{k\alpha-k\gamma}}\,dtd\rho$$

$$= A(k,\alpha)W_k^k(k\alpha-k\gamma;\, R, \theta)$$

(since $k\gamma-k\beta-1 = k-3$), and this implies (7.3).

† We note here that the case $k \geqslant 2$ of (7.3) is an improvement on the inequality (2.3.3) of (6), and may be used to effect a slight simplification in the arguments of Theorem 4 of that paper.

LEMMA 6. *Let $k \geqslant 1$ and $\sigma > 1$, and let W be defined by* (7.2). *Then*

$$\sup_{0<R<1}\{W_k^k(\sigma;\, R,\theta)\} \leqslant A(k,\sigma) \sup_{0<|u|\leqslant\pi}\left\{\frac{1}{u}\int_0^u G_k^k(\theta+v)\,dv\right\}. \tag{7.5}$$

Combining Lemmas 5 and 6 we obtain immediately the inequality (4.5). To prove (7.5) let

$$X(R,\theta) = \int_0^R (1-\rho)^{k-2}|\phi'(\rho e^{i\theta})|^k\,d\rho,$$

so that $X(R,\theta) \leqslant (1-R)^{-1}G_k^k(\theta)$. By integration by parts, we have

$$\int_0^R (1-\rho)^{k+\sigma-3}\frac{|\phi'(\rho e^{i\theta+it})|^k}{|1-\rho e^{it}|^\sigma}\,d\rho$$

$$= \left[X(\rho,\theta+t)\frac{(1-\rho)^{\sigma-1}}{|1-\rho e^{it}|^\sigma}\right]_0^R - \int_0^R X(\rho,\theta+t)\frac{\partial}{\partial\rho}\left\{\frac{(1-\rho)^{\sigma-1}}{|1-\rho e^{it}|^\sigma}\right\}d\rho.$$

Since

$$\left|\frac{\partial}{\partial\rho}\left\{\frac{(1-\rho)^{\sigma-1}}{|1-\rho e^{it}|^\sigma}\right\}\right| \leqslant \left|\frac{\partial}{\partial\rho}\left\{\frac{(1-\rho)^{\sigma-1}}{(1-\rho e^{it})^\sigma}\right\}\right| \leqslant A(\sigma)\left\{\frac{(1-\rho)^{\sigma-2}}{|1-\rho e^{it}|^\sigma} + \frac{(1-\rho)^{\sigma-1}}{|1-\rho e^{it}|^{\sigma+1}}\right\}$$

$$\leqslant A(\sigma)\frac{(1-\rho)^{\sigma-2}}{|1-\rho e^{it}|^\sigma},$$

we obtain

$$\int_0^R (1-\rho)^{k+\sigma-3}\frac{|\phi'(\rho e^{i\theta+it})|^k}{|1-\rho e^{it}|^\sigma}\,d\rho$$

$$\leqslant (1-R)^{\sigma-2}\frac{G_k^k(\theta+t)}{|1-Re^{it}|^\sigma} + A(\sigma)\int_0^R (1-\rho)^{\sigma-3}\frac{G_k^k(\theta+t)}{|1-\rho e^{it}|^\sigma}\,d\rho.$$

Integrating with respect to t, interchanging the order of integration in the second term on the right, and using the first result in Lemma 3, we obtain the required inequality.

Consider now the proof of Theorem 2. From Lemmas 2, 5, and 6 we have

$$\|H_{k,\alpha}\|_\lambda \leqslant A(k,\alpha)\|G_k\|_\lambda \tag{7.6}$$

for $\lambda > k \geqslant 1$ and $\alpha > \sup(1/k, 1/k')$. If now $\lambda > k \geqslant 2$, we have (7.6) for $\alpha > 1/k'$. If $\lambda > 2 > k$, then we have (7.6) for $\alpha > 1/k$, and so, in particular, for $\alpha = 2$. From (3.4) and (3.5) we have also

$$G_2 \leqslant A(\alpha)H_{2,\alpha+1} \leqslant A(k,\alpha)H_{k,\alpha},$$

and taking $\alpha = 2$ and combining this with the particular case of (7.6) already proved we obtain

$$\|G_2\|_\lambda \leqslant A(k,\lambda)\|G_k\|_\lambda. \tag{7.7}$$

Since also $\|H_{2,\alpha}\|_\lambda \leqslant A(\alpha,\lambda)\|G_2\|_\lambda$

for $\alpha > \frac{1}{2}$, and, by (3.3), $H_{k,\alpha} \leqslant H_{2,\alpha}$,

we obtain (7.6) for $\alpha > \frac{1}{2}$. Finally, if $2 \geqslant \lambda > k$, we argue as above, replacing the index 2 by an index l such that $\lambda > l > k$ and as close to λ as we please. We then obtain (7.6) for $\alpha > 1/\lambda$, and this completes the proof of the theorem.

We note that the argument which proves (7.7) gives also the more general result that $\|G_r\|_\lambda \leqslant A(k,r,\lambda)\|G_k\|_\lambda$

for $1 \leqslant k < r \leqslant \infty$, $\lambda > k$. This, too, is an 'averaged' Abelian theorem and asserts that on the average $G_r(\theta)$ does not exceed some multiple of $G_k(\theta)$, a result which is in general false for any particular θ [cf. (3.3)]. There are also results of this type for $\mathbf{G}$ and g; these can be proved by similar arguments.

8. We pass next to the proof of (4.3) and (5.8). In virtue of (5.13) and (5.7), it is enough to prove (5.8), i.e. the theorem:

THEOREM 3. *If* $1 \leqslant k < \infty$ *and* $\lambda > 0$, *then*

$$\|\phi\|_\lambda \leqslant A(k,\lambda)\|\mathbf{G}_k\|_\lambda. \tag{8.1}$$

An argument entirely similar to that of Theorem 2 gives

$$\|\mathbf{H}_{k,\alpha}\|_\lambda \leqslant A(k,\alpha,\lambda)\|\mathbf{G}_k\|_\lambda$$

for $\lambda > k \geqslant 1$ and $\alpha > \sup(1/k,\ 1/k')$. Combining this with (3.4) (with $r = \infty$), (3.5) (with $k = \infty$), and (5.7) we obtain immediately the result (8.1) for $\lambda > k \geqslant 1$.† It is therefore enough to prove (8.1) for $0 < \lambda \leqslant k$, and for this we use Minkowski's inequality.

If $0 < \lambda \leqslant k$, we have‡

$$\int_0^R (1-\rho)^{-2}\,d\rho\left\{\int_{-\pi}^{\pi}|\phi(\rho e^{i\theta})|^\lambda\,d\theta\right\}^{k/\lambda} \leqslant \left\{\int_{-\pi}^{\pi}d\theta\left(\int_0^R (1-\rho)^{-2}|\phi(\rho e^{i\theta})|^k\,d\rho\right)^{\lambda/k}\right\}^{k/\lambda}.$$

Writing $$Y(\rho) = \left\{\int_{-\pi}^{\pi}|\phi(\rho e^{i\theta})|^\lambda\,d\theta\right\}^{k/\lambda},$$

we obtain immediately

$$(1-R)\int_0^R (1-\rho)^{-2}Y(\rho)\,d\rho \leqslant \|\mathbf{G}_k\|_\lambda^k. \tag{8.2}$$

† This argument is curiously indirect. Essentially we prove (4.3) for $\lambda > k \geqslant 1$ and (5.8) for $0 < \lambda \leqslant k$.

‡ See, for example, Littlewood (**14**) Theorem 2 (7).

Now let $0 < \sigma < \frac{1}{2}$. Since $Y(\rho)$ is an increasing function, we have

$$\int_{1-2\sigma}^{1-\sigma} (1-\rho)^{-2} Y(\rho)\, d\rho \geqslant Y(1-2\sigma) \int_{1-2\sigma}^{1-\sigma} (1-\rho)^{-2}\, d\rho = \frac{1}{2\sigma} Y(1-2\sigma). \quad (8.3)$$

On the other hand, by (8.2),

$$\int_{1-2\sigma}^{1-\sigma} (1-\rho)^{-2} Y(\rho)\, d\rho \leqslant \int_{0}^{1-\sigma} \leqslant \frac{1}{\sigma} \|\mathbf{G}_k\|_\lambda^k, \quad (8.4)$$

and combining (8.3) and (8.4) we get

$$Y(1-2\sigma) \leqslant 2\|\mathbf{G}_k\|_\lambda^k.$$

Making $\sigma \to 0$ we obtain now (8.1) for $0 < \lambda \leqslant k$, and this completes the proof of Theorem 3.

9. We turn now to the proofs of the 'boundary' results of § 6, and begin with that of (6.7), which we combine with a new proof of the case $\lambda \leqslant k$ of (5.11). We state both results as

THEOREM 4. *Let*

$$j_{k,\alpha,\delta}(\theta) = \left\{ \sum_1^\infty \frac{|\tau_n^\alpha(\theta)|^k}{n(\log(n+1))^{k\delta}} \right\}^{1/k}$$

(so that $j_{k,\alpha,0} = h_{k,\alpha}$*). Then, if* $k \geqslant 2$ *and* $\lambda > 0$*, and if*

either $\quad \alpha = \sup(1/k', 1/\lambda) \quad$ *and* $\quad \delta = \sup(1/k, 1/\lambda)$

or $\quad \alpha > \sup(1/k', 1/\lambda) \quad$ *and* $\quad \delta = 0,$

$$\|j_{k,\alpha,\delta}\|_\lambda \leqslant A(k,\alpha,\lambda)\|\phi\|_\lambda.$$

We require first the lemma:

LEMMA 7. *Let* $\beta > 0$ *and* $\gamma \geqslant 1$*. Then, for all* δ*,*

$$\int_0^1 (1-\rho)^{\beta-1} \left(\log \frac{2}{1-\rho}\right)^\delta \rho^\gamma\, d\rho \geqslant A(\beta,\delta)\gamma^{-\beta}\{\log(\gamma+1)\}^\delta.$$

We have

$$\int_0^1 \geqslant \int_{1-1/\gamma}^{1-1/2\gamma} \geqslant A(\beta,\delta)\gamma^{1-\beta}\{\log(\gamma+1)\}^\delta \int_{1-1/\gamma}^{1-1/2\gamma} \rho^\gamma\, d\rho \geqslant A(\beta,\delta)\gamma^{-\beta}\{\log(\gamma+1)\}^\delta.$$

Our proof of Theorem 4 depends on a majorant of j of the usual type. Let

$$J_{k,l}(\sigma,\delta;\theta) = J_{k,l}(\sigma,\delta;\theta;\phi)$$

$$= \left[\int_0^1 (1-\rho)^{k(\sigma+1-1/k-1/l)} \left(\log \frac{2}{1-\rho}\right)^{-k\delta} \left(\int_{-\pi}^{\pi} \frac{|\phi'(\rho e^{i\theta+it})|^l}{|1-\rho e^{it}|^{l\sigma}}\, dt \right)^{k/l} d\rho \right]^{1/k}.$$

Then we have

LEMMA 8. *If $k \geqslant 2$ and $0 < l \leqslant k$, then, for all δ,*

$$j_{k,\alpha,\delta}(\theta) \leqslant A(k,l,\alpha,\delta) J_{kl}(\alpha,\delta;\theta) \quad (0 < l \leqslant k';\ \alpha > 1/l - 1)$$

and

$$j_{k,\alpha,\delta}(\theta) \leqslant A(k,l,\alpha,\delta) J_{kl}(\alpha + 1/k + 1/l - 1, \delta;\theta) \quad (k' \leqslant l \leqslant k;\ \alpha > -1/k).$$

This follows easily from Lemma 4 and Lemma 7 [cf. Lemma 5; in Lemma 4 take $\gamma = 0$ or $\beta = 0$ according as $0 < l \leqslant k'$ or $k' \leqslant l \leqslant k$].

We require also two preliminary results which depend in turn on the case $\lambda = k = 2$ of the theorem. For this case there are available several arguments which are not applicable to the general result, and we therefore deal especially with this case. We use the formula

$$j_{2,\alpha,\delta}^2(\theta) \leqslant A(\alpha,\delta) \int_0^1 (1-\rho)^{2\alpha} \left(\log \frac{2}{1-\rho}\right)^{-2\delta} d\rho \int_{-\pi}^{\pi} \frac{|\phi'(\rho e^{i\theta + it})|^2}{|1-\rho e^{it}|^{2\alpha}}\, dt, \quad (9.1)$$

which is the case $k = l = 2$ of Lemma 8. We note, however, that in this case Lemma 4 (on which the proof of Lemma 8 depends) reduces to Parseval's theorem, so that (9.1) is considerably more elementary than the general result of Lemma 8.

We prove now

LEMMA 9.† *If either $\alpha = \delta = \frac{1}{2}$ or $\alpha > \frac{1}{2}$ and $\delta = 0$, then*

$$\|j_{2,\alpha,\delta}\|_2 \leqslant A(\alpha)\|\phi\|_2.$$

Consider the first. By (9.1) we have

$$\|j_{2,\frac{1}{2},\frac{1}{2}}\|_2^2 \leqslant A \int_{-\pi}^{\pi} d\theta \int_0^1 (1-\rho)\left(\log \frac{2}{1-\rho}\right)^{-1} d\rho \int_{-\pi}^{\pi} \frac{|\phi'(\rho e^{i\theta+it})|^2}{|1-\rho e^{it}|}\, dt$$

$$= A \int_0^1 (1-\rho)\left(\log \frac{2}{1-\rho}\right)^{-1} d\rho \int_{-\pi}^{\pi} \frac{dt}{|1-\rho e^{it}|} \int_{-\pi}^{\pi} |\phi'(\rho e^{i\theta+it})|^2\, d\theta.$$

The innermost integral here is independent of t and is equal to

$$2\pi \sum_1^\infty n^2 |c_n|^2 \rho^{2n-2}$$

(by Parseval's theorem), while

$$\int_{-\pi}^{\pi} \frac{dt}{|1-\rho e^{it}|} \leqslant A \log \frac{2}{1-\rho}.$$

† Zygmund (**26**) gives an alternative proof of the first case of Lemma 9.

Hence

$$\|j_{2,\frac{1}{2},\frac{1}{2}}\|_2^2 \leqslant A\sum_1^\infty n^2|c_n|^2\int_0^1(1-\rho)\rho^{2n-2}\,d\rho \leqslant A\sum_1^\infty |c_n|^2 \leqslant A\|\phi\|_2^2,$$

as required. The proof of the second case is similar, and we omit it.

We can now prove our auxiliary results.

LEMMA 10. *Let*

$$p_\lambda(\theta) = p_\lambda(\theta;\phi) = \sup_\rho\left\{\left(\log\frac{2}{1-\rho}\right)^{-1}\int_{-\pi}^{\pi}\frac{|\phi(\rho e^{i\theta+it})|^\lambda}{|1-\rho e^{it}|}\,dt\right\}^{1/\lambda}.$$

If $\lambda > 0$, *then* $\qquad \|p_\lambda\|_\lambda \leqslant A(\lambda)\|\phi\|_\lambda.$

By the usual reduction to the case of a zero-free ϕ, we may suppose $\lambda = 2$. Then, by Parseval's theorem applied to the function

$$\frac{\phi(ze^{i\theta})}{(1-z)^{\alpha+1}} = \sum_0^\infty E_n^\alpha\sigma_n^\alpha(\theta)z^n, \tag{9.2}$$

we have†

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{|\phi(\rho e^{i\theta+it})|^2}{|1-\rho e^{it}|}\,dt &= 2\pi\sum_0^\infty (E_n^{-\frac{1}{2}})^2|\sigma_n^{-\frac{1}{2}}(\theta)|^2\rho^{2n}\\ &\leqslant A\sum_1^\infty n^{-1}|\tau_n^{\frac{1}{2}}(\theta)|^2\rho^{2n}+A\sum_0^\infty (n+1)^{-1}|\sigma_n^{\frac{1}{2}}(\theta)|^2\rho^{2n}\\ &\leqslant A\sum_1^\infty n^{-1}|\tau_n^{\frac{1}{2}}(\theta)|^2\rho^{2n}+A\left(\log\frac{2}{1-\rho}\right)\sup_n|\sigma_n^{\frac{1}{2}}(\theta)|^2.\end{aligned}$$

Since also $\qquad \rho^{2n}\left(\log\frac{2}{1-\rho}\right)^{-1} \leqslant \frac{A}{\log(n+1)},$

we have now $\qquad p_2^2(\theta) \leqslant Aj_{2,\frac{1}{2},\frac{1}{2}}^2(\theta)+A\sup_n|\sigma_n^{\frac{1}{2}}(\theta)|^2,$

and the result now follows from Lemma 9 and a maximal theorem of Hardy and Littlewood (**9**) [actually the case $k = \infty$ of (5.13)].

LEMMA 11. *Let*

$$P_{\lambda,\mu}(\theta) = P_{\lambda,\mu}(\theta;\phi) = \sup_\rho\left\{(1-\rho)^{\mu-1}\int_{-\pi}^{\pi}\frac{|\phi(\rho e^{i\theta+it})|^\lambda}{|1-\rho e^{it}|^\mu}\,dt\right\}^{1/\lambda}.$$

If $\lambda > 0$ *and* $\mu > 1$, *then*

$$\|P_{\lambda,\mu}\|_\lambda \leqslant A(\lambda,\mu)\|\phi\|_\lambda.$$

The proof of this follows on almost identical lines to that of Lemma 10, and we omit it.

† We recall that, for $\alpha > 0$,

$$\tau_n^\alpha = -\alpha(\sigma_n^\alpha-\sigma_n^{\alpha-1}).$$

It is enough now to deal with J, where l is at our disposal, and we distinguish two cases. For convenience we treat these in lemmas.

LEMMA 12. *If $\lambda \geqslant k \geqslant 2$ and either $\sigma = \delta = 1/k$ or $\sigma > 1/k$ and $\delta = 0$, then*

$$\|J_{k,k}\|_\lambda \leqslant A(k,\lambda,\sigma)\|\phi\|_\lambda. \tag{9.3}$$

It is proved in § 3.6 of (4) that if $\lambda \geqslant k \geqslant 1$, $\sigma > 1/k$, $\delta = 0$, then

$$\|J_{k,k}\|_\lambda \leqslant A(k,\lambda,\sigma)\|g_k\|_\lambda.$$

A similar argument (using the second result of Lemma 3 in place of the first) shows that this result also holds for $\sigma = \delta = 1/k$. The lemma therefore follows with the aid of (5.1).

LEMMA 13. *If $k \geqslant 2$, $0 < \lambda < k$, and either $\sigma = \delta = 1/\lambda$ or $\sigma > 1/\lambda$ and $\delta = 0$, then*

$$\|J_{k,\lambda}\|_\lambda \leqslant A(k,\lambda,\sigma)\|\phi\|_\lambda.$$

By the usual reduction, we may suppose that ϕ does not vanish in $|z| < 1$. Write then $\phi = \psi^{k/\lambda}$, so that ψ is regular and of class H^k in $|z| < 1$, and

$$|\phi'|^\lambda = A(k,\lambda)|\psi|^{k-\lambda}|\psi'|^\lambda.$$

By Hölder's inequality with indices $k/(k-\lambda)$ and k/λ, we have

$$\int_{-\pi}^{\pi} \frac{|\phi'(\rho e^{i\theta+it})|^\lambda}{|1-\rho e^{it}|^\mu}\,dt \leqslant A(k,\lambda)\left\{\int_{-\pi}^{\pi} \frac{|\psi(\rho e^{i\theta+it})|^k}{|1-\rho e^{it}|^\mu}\,dt\right\}^{(k-\lambda)/k}\left\{\int_{-\pi}^{\pi} \frac{|\psi'(\rho e^{i\theta+it})|^k}{|1-\rho e^{it}|^\mu}\,dt\right\}^{\lambda/k} \tag{9.4}$$

for any μ. It follows now (with $\mu = 1$) that

$$J^\lambda_{k,\lambda}(1/\lambda, 1/\lambda; \theta; \phi) \leqslant A(k,\lambda)p_k^{k-\lambda}(\theta;\psi)J^\lambda_{k,k}(1/k, 1/k; \theta; \psi).$$

Integrating over $(-\pi,\pi)$, applying Hölder's inequality with indices $k/(k-\lambda)$ and k/λ to the right-hand side, and using Lemma 10 and the case $\lambda = k$ of Lemma 12, we obtain the first case of the lemma. Similarly, taking $\mu = \lambda\sigma$ in (9.4), we obtain

$$J^\lambda_{k,\lambda}(\sigma, 0; \theta; \phi) \leqslant A(k,\lambda)P^{k-\lambda}_{k,\lambda\sigma}(\theta;\psi)J^\lambda_{k,k}\left(\frac{\lambda\sigma}{k}, 0; \theta; \psi\right),$$

and the second case now follows by the same argument as before, save that Lemma 11 is used in place of Lemma 10.

From Lemmas 8, 12, and 13, we obtain the result of the theorem.

10. We arrive finally at the proof of the case (i) of (6.1), i.e.

THEOREM 5. *If $0 < \lambda \leqslant 1$, then*

$$\left\|\sup_{n\geqslant 1}\{|\sigma_n^{1/\lambda-1}(\theta)|/\log^{1/\lambda}(n+1)\}\right\|_\lambda \leqslant A(\lambda)\|\phi\|_\lambda.$$

By (9.2), we have†

$$\rho^n E_n^{1/\lambda-1}|\sigma_n^{1/\lambda-1}(\theta)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\phi(\rho e^{i\theta+it})e^{-nti}}{(1-\rho e^{it})^{1/\lambda}}\,dt\right| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\phi(\rho e^{i\theta+it})|}{|1-\rho e^{it}|^{1/\lambda}}\,dt \tag{10.1}$$

for $\rho < 1$. Multiplying both sides by

$$(1-\rho)^{1/\lambda-1}\left(\log\frac{2}{1-\rho}\right)^{-1/\lambda}$$

and then taking $\rho = 1-1/(n+1)$, we see easily that

$$\sup_{n\geqslant 1}\{|\sigma_n^{1/\lambda-1}(\theta)|\,/\log^{1/\lambda}(n+1)\} \leqslant A(\lambda)r_\lambda(\theta), \tag{10.2}$$

where

$$r_\lambda(\theta) = \sup_\rho\left\{(1-\rho)^{1/\lambda-1}\left(\log\frac{2}{1-\rho}\right)^{-1/\lambda}\int_{-\pi}^{\pi}\frac{|\phi(\rho e^{i\theta+it})|}{|1-\rho e^{it}|^{1/\lambda}}\,dt\right\},$$

and the case $\lambda = 1$ of the theorem is now an immediate consequence of (10.2) and Lemma 10.

Suppose then that $0 < \lambda < 1$. Using the case $0 < \lambda < k = 2$ of Theorem 4, and applying once again the argument of Lemma 10, we obtain that

$$\int_{-\pi}^{\pi}\sup_\rho\left\{(1-\rho)^{2\lambda-1}\left(\log\frac{2}{1-\rho}\right)^{-2\lambda}\int_{-\pi}^{\pi}\frac{|\phi(\rho e^{i\theta+it})|^{2\eta}}{|1-\rho e^{it}|^{2\lambda}}\,dt\right\}^{\frac{1}{2}\lambda}d\theta$$

$$\leqslant A(\lambda)\int_{-\pi}^{\pi}|\phi(e^{i\theta})|^{\lambda\eta}\,d\theta$$

for $0 < \lambda < 2$ and for any $\eta > 0$. Choosing $\eta = \frac{1}{2}$, and then replacing $\frac{1}{2}\lambda$ by λ, where now $0 < \lambda < 1$, we obtain that

$$\|r_\lambda\|_\lambda \leqslant A(\lambda)\|\phi\|_\lambda, \tag{10.3}$$

and again the result follows from (10.2).‡‡

11. There is an alternative proof of Theorem 5 which uses only the case $\lambda = k = 2$ of Theorem 4 (and so avoids the use of the general case of Lemma 4).§

† Here the oscillating factor e^{-nti} in the integral representation for $\sigma_n^{1/\lambda-1}(\theta)$ disappears.

‡ There are slightly easier methods of deducing the case $0 < \lambda < 1$ from Theorem 4 [see Sunouchi (**21**) Lemma 5], but I have selected one which makes explicit the reliance on the crude estimate of (10.1).

§ On the other hand, it introduces an argument used by Zygmund (**27**) and the author (**4**) in the proof of the various cases of (5.11).

We require a lemma which is perhaps of interest in itself. It provides a substitute for regular functions for the case $\lambda = 1$ of Lemma 2.

LEMMA 14. *Let*

$$q_\lambda(\theta) = q_\lambda(\theta; \phi) = \sup_\rho \left\{ \left(\log \frac{2}{1-\rho} \right)^{-1} \sup_{0<|u|\leqslant\pi} \left(\frac{1}{u} \int_0^u |\phi(\rho e^{i\theta+iv})|^\lambda \, dv \right) \right\}^{1/\lambda}.$$

If $\lambda > 0$, *then* $$\|q_\lambda\|_\lambda \leqslant A(\lambda)\|\phi\|_\lambda.$$

Let $$Z(\rho, u) = \frac{1}{u} \int_0^u |\phi(\rho e^{i\theta+iv})|^\lambda \, dv.$$

When $0 \leqslant |v| \leqslant 1-\rho$, the point $z = \rho e^{i\theta+iv}$ lies in the kite-shaped region $\Omega(\theta)$ for some fixed η, so that

$$\sup_{0\leqslant|u|\leqslant 1-\rho} Z(\rho, u) \leqslant \Phi^\lambda(\theta). \tag{11.1}$$

On the other hand, if $\pi \geqslant |u| \geqslant 1-\rho$ and $0 \leqslant |v| \leqslant |u|$, then

$$|u| \geqslant \tfrac{1}{2}(1-\rho)+\tfrac{1}{2}|u| \geqslant A\{(1-\rho)^2+u^2\}^{\frac{1}{2}} \geqslant A|1-\rho e^{iu}| \geqslant A|1-\rho e^{iv}|,$$

so that $$\sup_{1-\rho\leqslant|u|\leqslant\pi} Z(\rho, u) \leqslant A \int_{-\pi}^{\pi} \frac{|\phi(\rho e^{i\theta+iv})|^\lambda}{|1-\rho e^{iv}|} \, dv. \tag{11.2}$$

From (11.1) and (11.2) we have now

$$q_\lambda^\lambda(\theta) \leqslant \Phi^\lambda(\theta)+Ap_\lambda^\lambda(\theta),$$

and the result then follows from (5.5) and Lemma 10.

Consider now the proof of Theorem 5. By Lemma 10 and (10.2), it is enough to prove (10.3) for $0 < \lambda < 1$. And, by the usual reduction, we may suppose that $\phi \neq 0$ in $|z| < 1$.

Since $|\phi|^\lambda$ is subharmonic in $|z| < 1$,

$$|\phi(\rho e^{i\theta})|^\lambda \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} |\phi(\rho^{\frac{1}{2}} e^{i\theta+it})|^\lambda P(\rho^{\frac{1}{2}}, t) \, dt,$$

where P is Poisson's kernel.† Since

$$P(\rho^{\frac{1}{2}}, t) = \frac{1-\rho}{|1-\rho^{\frac{1}{2}}e^{it}|^2} \leqslant A \frac{1-\rho}{|1-\rho e^{it}|^2},$$

Lemma 3 now gives

$$|\phi(\rho e^{i\theta+it})|^\lambda \leqslant A \frac{|1-\rho e^{it}|}{1-\rho} \sup_{0<|u|\leqslant\pi} \left(\frac{1}{u} \int_0^u |\phi(\rho^{\frac{1}{2}} e^{i\theta+iv})|^\lambda \, dv \right).$$

† See, for example, Littlewood (**14**) § 21.

Hence

$$r_\lambda(\theta) \leqslant A(\lambda)\sup_\rho\left\{\left(\log\frac{2}{1-\rho}\right)^{-1/\lambda}\left(\int_{-\pi}^{\pi}\frac{|\phi(\rho e^{i\theta+iu})|^\lambda}{|1-\rho e^{iu}|}\,du\right)\times\right.$$

$$\left.\times\sup_{0<|u|\leqslant\pi}\left(\frac{1}{u}\int_0^u|\phi(\rho^{\frac{1}{2}}e^{i\theta+iv})|^\lambda\,dv\right)^{1/\lambda-1}\right\} = A(\lambda)p_\lambda^\lambda(\theta)q_\lambda^{1-\lambda}(\theta).$$

Raising both sides to the λth power, integrating with respect to θ over $(-\pi,\pi)$, applying Hölder's inequality with indices $1/\lambda$ and $1/(1-\lambda)$, and using Lemmas 10 and 14, we obtain (10.3) as required.

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EXPANSIONS OF WHITTAKER'S FUNCTION

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1. Introduction

WHITTAKER'S confluent hypergeometric function (**1**) is defined, when $2m$ is not a negative integer, by the equation

$$M_{k,m}(z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}+m} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+m-k)_n}{n!(2m+1)_n} z^n. \tag{1}$$

Slater [(**2**) (1.5)] has shown that this function can be expressed in terms of a series of Bessel functions. Her result can be written

$$M_{k,m}(z) = z^{\frac{1}{2}-k} 4^{m+k} \Gamma(m+k) \sum_{n=0}^{\infty} (-)^n \frac{(2k)_n (2m+2k)_n}{n!(2m+1)_n} (m+k+n) I_{m+k+n}(\tfrac{1}{2}z). \tag{2}$$

As is well known, the function $M_{k,m}(z)$ is of considerable importance in nuclear scattering problems (**3**). It expresses, apart from a certain factor, the regular radial wave-function of a particle moving in a Coulomb field of force. Expansions of $M_{k,m}(z)$ in series of Bessel functions are very useful for computing the numerical values of the wave functions. The expansion (2) is not convenient for this purpose because in a repulsive Coulomb field the parameter k is purely imaginary. Convenient expansions have been developed by Morse and Abramowitz [(**3**) XVIII (2.1), (2.7)]. The object of this paper is to present an expansion of $M_{k,m}(z)$ which generalizes the expansions of Slater, Morse, and Abramowitz and, moreover, is capable of furnishing other expansions.

2. The expansion of $M_{k,m}(z)$

This function can be represented by the contour integral

$$M_{k,m}(z) = -\frac{\Gamma(2m+1)}{2\pi i} e^{-\frac{1}{2}z} z^{\frac{1}{2}+m} \int_{\infty}^{(0+)} (-t)^{-2m-1} \left(1+\frac{z}{t}\right)^{-\frac{1}{2}-m+k} e^{-t}\, dt, \tag{3}$$

where the path of integration starts at infinity on the real axis, encircles the origin in the positive direction, and returns to plus infinity; on the contour $|z| < |t|$.

Quart. J. Math. Oxford (2) 10 (1959) 202-5.

Consider the factor $(1+z/t)^{-\frac{1}{2}-m+k}$ in the integrand and write

$$(1+z/t)^{-\frac{1}{2}-m+k} = (1+z/t)^{-\frac{1}{2}-2m+\lambda}(1+z/t)^{m+k-\lambda},$$

where λ is any number. Expanding the factor $(1+z/t)^{m+k-\lambda}$ by the binomial theorem into a uniformly convergent series, we have

$$\left(1+\frac{z}{t}\right)^{-\frac{1}{2}-m+k} = \left(1+\frac{z}{t}\right)^{-\frac{1}{2}-2m+\lambda} \sum_{n=0}^{\infty} \frac{(\lambda-m-k)_n}{n!}\left(-\frac{z}{t}\right)^n. \tag{4}$$

Substituting (4) in (3) and integrating term by term, we have

$$M_{k,m}(z) = -\frac{\Gamma(2m+1)}{2\pi i} e^{-\frac{1}{2}z} z^{\frac{1}{2}+m} \sum_{n=0}^{\infty} \frac{(\lambda-m-k)_n}{n!} z^n \times$$

$$\times \int_{\infty}^{(0+)} (-t)^{-2m-1-n}\left(1+\frac{z}{t}\right)^{-\frac{1}{2}-2m+\lambda} e^{-t}\, dt.$$

From (3), it follows that

$$M_{k,m}(z) = \sum_{n=0}^{\infty} \frac{(\lambda-m-k)_n}{n!\,(2m+1)_n} z^{\frac{1}{2}n} M_{\lambda-m+\frac{1}{2}n,m+\frac{1}{2}n}(z). \tag{5}$$

Now, applying (2) to the function $M_{\lambda-m+\frac{1}{2}n,m+\frac{1}{2}n}(z)$, we have

$$z^{\frac{1}{2}n} M_{\lambda-m+\frac{1}{2}n,m+\frac{1}{2}n}(z) = 4^{\lambda+n}\Gamma(\lambda+n) z^{\frac{1}{2}-\lambda+m} \times$$

$$\times \sum_{\nu=0}^{\infty} (-)^{\nu} \frac{(2\lambda-2m+n)_\nu (2\lambda+2n)_\nu}{\nu!\,(2m+n+1)_\nu} (\lambda+n+\nu) I_{\lambda+n+\nu}(\tfrac{1}{2}z). \tag{6}$$

Substituting (6) in (5) and arranging the resulting series in a series of Bessel functions, we obtain

$$M_{k,m}(z) = 4^{\lambda}\Gamma(\lambda+1) z^{\frac{1}{2}+m-\lambda} \sum_{n=0}^{\infty} a_n I_{\lambda+n}(\tfrac{1}{2}z), \tag{7}$$

where

$$a_n = (-)^n \frac{(2\lambda-2m)_n (2\lambda)_n (\lambda+n)}{n!\,(2m+1)_n (\lambda)} {}_3F_2\left[\begin{matrix} -n,\ \lambda-m-k,\ 2\lambda+n; \\ 2\lambda-2m,\ \lambda+\tfrac{1}{2}; \end{matrix} 1\right]. \tag{8}$$

Since $a_{n+1} I_{\lambda+n+1}/a_n I_{\lambda+n} \to 0$ as $n \to \infty$, the arrangement is permissible. The expansion (7) is valid when λ is not a negative integer. It generalizes Slater's expansion, which corresponds to (7) with $\lambda = m+k$.

The expressions a_n can be presented in the convenient alternative way,

$$a_0 = 1, \qquad a_1 = -4k(\lambda+1)/(2m+1),$$

$$a_2 = \{4k^2(\lambda+1)+(\lambda-m)(1+2m)\}(\lambda+2)/(2m+1)(m+1),$$

and in general

$$na_n = \frac{(\lambda+n)}{(2m+n)}\left\{-4ka_{n-1}+\frac{(2\lambda+n-2)(2\lambda-2m+n-2)}{(\lambda+n-2)}a_{n-2}\right\} \quad (9)$$

This is useful for computing the numerical values of the coefficients of the expansion.

Evidently (7) together with (8) or (9) could be used to write down suitable Bessel-series expansions for functions of the type $M_{k,m}(z)$. Examples of these functions are the incomplete gamma function, the error function, Fresnel's integrals, and Weber's parabolic cylinder function. In particular, when $k = 0$,

$$M_{0,m}(z) = 2^{2m}\Gamma(m+1)z^{\frac{1}{2}}I_m(\tfrac{1}{2}z),$$

so that we can readily deduce from (7) and (9) the expansion given by Watson [(**4**) 139] of a Bessel function as a series of Bessel functions.

3. The regular radial Coulomb wave-function

This function, in normalized form, is expressible in terms of Whittaker's function (**3**) by

$$F_L(\eta,\rho) = C_L(\eta)(2i)^{-L-1}M_{i\eta,L+\frac{1}{2}}(2i\rho), \quad (10)$$

where

$$C_L(\eta) = \frac{2^L}{(2L+1)!}\left\{(1^2+\eta^2)(2^2+\eta^2)\ldots(L^2+\eta^2)\frac{2\pi\eta}{e^{2\pi\eta}-1}\right\}^{\frac{1}{2}},$$

L is the orbital angular-momentum quantum number, $\eta = ZZ'e^2/\hbar v$ and $\rho = \mu vr/\hbar$; v is the relative velocity of the colliding particles, r is the distance between them, μ is the reduced mass and Ze, $Z'e$ are their charges.

Using (7) and (9), we can expand $F_L(\eta,\rho)$ as

$$F_L(\eta,\rho) = C_L(\eta)2^{\lambda}\Gamma(\lambda+1)\rho^{L+1-\lambda}\sum_{n=0}^{\infty}b_n J_{\lambda+n}(\rho), \quad (11)$$

where

$$b_0 = 1, \qquad b_1 = 2\eta(\lambda+1)/(L+1),$$

$$b_2 = \{4\eta^2(\lambda+1)-(2\lambda-2L-1)(L+1)\}(\lambda+2)/(L+1)(2L+3),$$

$$nb_n = \frac{(\lambda+n)}{(2L+1+n)}\left\{4\eta b_{n-1}-\frac{(2\lambda+n-2)(2\lambda+n-2L-3)}{(\lambda+n-2)}b_{n-2}\right\}. \quad (12)$$

This generalizes the expansions given by Morse and Abramowitz. Morse's expansion corresponds to (11) and (12) with $\lambda = L+\frac{1}{2}$, and that of Abramowitz is obtained by putting $\lambda = 0$. Taking $\lambda = 2L+1$, one also has the simple expansion

$$F_L(\eta,\rho) = C_L(\eta)2^{2L+1}(2L+1)!\,\rho^{-L}\sum_{n=0}^{\infty}c_n J_{2L+1+n}(\rho), \quad (13)$$

where

$$c_0 = 1, \qquad c_1 = 4\eta, \qquad c_2 = 8\eta^2 - 2L - 1,$$

$$nc_n = 4\eta c_{n-1} - (4L+n)c_{n-2}. \tag{14}$$

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APPROXIMATE VALUES OF LAMÉ FUNCTIONS

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Summary

In this paper functions are derived which are approximate solutions of Lamé's differential equation when the difference between two of the singularities is small. An approximate value of the characteristic parameter is obtained, and the corresponding solutions are expressed in terms of associated Legendre functions.

1. Notation

Lamé's differential equation in algebraic form is

$$4x^{\frac{1}{2}}(x-1)^{\frac{1}{2}}(x-k^2)^{\frac{1}{2}}\frac{d}{dx}x^{\frac{1}{2}}(x-1)^{\frac{1}{2}}(x-k^2)^{\frac{1}{2}}\frac{du}{dx}-\{n(n+1)x-h\}u=0 \quad (1.1)$$

with $k^2<1$. Writing $x=k^2\mathrm{sn}^2(\alpha,k)$ we have the Jacobian form

$$\frac{d^2u}{d\alpha^2}-\{n(n+1)k^2\mathrm{sn}^2\alpha-h\}u=0. \quad (1.2)$$

Writing $x=\rho^2$ we have the equation

$$(\rho^2-1)(\rho^2-k^2)\frac{d^2u}{d\rho^2}+\rho(2\rho^2-1-k^2)\frac{du}{d\rho}-\{n(n+1)\rho^2-h\}u=0. \quad (1.3)$$

The linearized differential equation for the velocity potential of a delta wing at supersonic speeds expressed in terms of hyperboloidoconal coordinates r, μ, ν has solutions (**3**)

$$r^nE_n(\rho_1)F_n(\rho_2),$$

where E_n and F_n are first and second solutions respectively of (1.3). Here $\rho_1=\nu\tan\gamma$, $\rho_2=\mu\tan\gamma$, $k^2=1-\cot^2m/\cot^2\gamma$, m being the Mach angle and γ the semi-apex angle of the delta wing. As the Mach angle decreases, k diminishes, and an approximation in which k^4 and higher powers of k may be neglected becomes possible.

Ince (**2**) denotes the $2n+1$ Lamé functions which are solutions of (1.2) as

$$\mathrm{Ec}_n^m(\alpha)\quad(m=0,1,\ldots,n),\qquad \mathrm{Es}_n^m(\alpha)\quad(m=1,2,\ldots,n),$$

the corresponding values of the auxiliary parameter h being denoted by a_n^m and b_n^m respectively. When using algebraic functions, it is convenient

to denote the solutions of (1.3) as $\mathscr{E}c_n^m(\rho)$ and $\mathscr{E}s_n^m(\rho)$, these solutions being respectively even and odd functions of ρ.

It is well known that the limits of a_n^m and b_n^m as $k \to 0$ in (1.3) are each m^2, and the Lamé functions which are solutions of (1.3) degenerate into associated Legendre functions: that is

$$\lim_{k\to 0} \mathscr{E}c_n^m(\rho) = \lim_{k\to 0} \mathscr{E}s_n^m(\rho) = P_n^m\{\sqrt{(1-\rho^2)}\}. \tag{1.4}$$

Lamé functions are usually undefined to the extent of a constant factor. The definition is completed by (1.4) if we specify the coefficient of the highest power of ρ as being the same as that in the corresponding associated Legendre function and independent of k.

Expansions of Lamé functions in a series of associated Legendre functions have been obtained by Erdélyi (**1**), and it is possible that some of the results obtained in this paper are deducible from Erdélyi's work.

Since $P_n^m\{\sqrt{(1-\rho^2)}\}$ has a zero of order m at the origin, it is easily seen from Stieltjes' theorem (**4**) that $\mathscr{E}c_n^m(\rho)$ and $\mathscr{E}s_n^m(\rho)$ have each m zeros in the interval $-k < \rho \leqslant k$, the interval being closed at one end to allow for a zero of the factor $(\rho^2-k^2)^{\frac{1}{2}}$ when it occurs.

The corresponding Lamé functions of the second kind may be denoted by the symbols $\mathscr{F}c_n^m(\rho)$ and $\mathscr{F}s_n^m(\rho)$ and defined by the equations

$$\mathscr{F}c_n^m(\rho) = \frac{(n+m)!}{(n-m)!}\mathscr{E}c_n^m(\rho)\int_t^\infty \frac{[\mathscr{E}c_n^m\{\sqrt{(1-s^2)}\}]^{-2}\,ds}{(s^2-1)^{\frac{1}{2}}(s^2-1+k^2)^{\frac{1}{2}}}, \tag{1.5}$$

$$\mathscr{F}s_n^m(\rho) = \frac{(n+m)!}{(n-m)!}\mathscr{E}s_n^m(\rho)\int_t^\infty \frac{[\mathscr{E}s_n^m\{\sqrt{(1-s^2)}\}]^{-2}\,ds}{(s^2-1)^{\frac{1}{2}}(s^2-1+k^2)^{\frac{1}{2}}}, \tag{1.6}$$

where $t = \sqrt{(1-\rho^2)}$. From these definitions it follows that

$$\lim_{k\to 0} \mathscr{F}c_n^m(\rho) = \lim_{k\to 0} \mathscr{F}s_n^m(\rho) = Q_n^m\{\sqrt{(1-\rho^2)}\}. \tag{1.7}$$

2. Approximation to the characteristic h

Consider the polynomial Lamé functions of even order $2n$ and even rank $2m$. These functions $\mathscr{E}c_{2n}^{2m}(\rho)$ are solutions of the differential equation

$$(\rho^2-1)(\rho^2-k^2)\frac{d^2u}{d\rho^2} + \rho(2\rho^2-1-k^2)\frac{du}{d\rho} - \{2n(2n+1)\rho^2-h\}u = 0: \tag{2.1}$$

that is, with $t^2 = 1-\rho^2$,

$$(t^2-1)(t^2-1+k^2)\frac{d^2u}{dt^2} + t(2t^2-2+k^2)\frac{du}{dt} - \{2n(2n+1)(t^2-1)+h\}u = 0. \tag{2.2}$$

Let a solution of (2.1) be

$$u = \sum_{r=0}^{n} \frac{\Gamma(\frac{1}{2}-2n)}{\Gamma(\frac{1}{2}-2n+r)r!} b_r \rho^{2n-2r}$$

and let b_0 be independent of k. Substituting in (2.1) we obtain the recurrence relation

$$b_{r+1} = \{(n-r)^2(1+k^2)-\tfrac{1}{4}h\}b_r + k^2 r(n+\tfrac{1}{2}-r)(n+1-r)(2n+\tfrac{1}{2}-r)b_{r-1}. \tag{2.3}$$

Let c_r be the value of b_r when $k = 0$. Then c_r satisfies a two-term recurrence relation, and we have

$$c_r = \prod_{s=0}^{r-1} \{(n-s)^2-\tfrac{1}{4}h\}b_0.$$

In this case the roots of the characteristic equation for h, found by putting $c_{n+1} = 0$, are

$$h = 4m^2 \quad (m = 0, 1, \ldots, n).$$

For a specific value of m,

$$c_r = \frac{(n-m)!\,(n+m)!}{(n-m-r)!\,(n+m-r)!} b_0.$$

It is evident that $c_r = 0$ when $r > n-m$.

If we write $v(\rho)$ for the value of $u(\rho)$ when $k = 0$ and $h = 4m^2$, (2.2) becomes the associated Legendre equation and, if

$$b_0 = \frac{(-1)^n(4n)!}{2^{2n}(2n)!\,(2n-2m)!},$$

then $$v(\rho) = P_{2n}^{2m}(t) = P_{2n}^{2m}\{\sqrt{(1-\rho^2)}\}.$$

We now assume that h can be expanded in powers of k^2 and write

$$h = 4m^2 + 4\alpha k^2 + O(k^4). \tag{2.4}$$

Let the coefficients b_r in (2.3) be of the form

$$b_r = c_r + k^2 d_r + O(k^4),$$

and $$u(\rho) = v(\rho) + k^2 w(\rho) + O(k^4),$$

so that $$w(\rho) = \sum_{r=0}^{n} \frac{\Gamma(\frac{1}{2}-2n)}{\Gamma(\frac{1}{2}-2n+r)r!} d_r \rho^{2n-2r}.$$

In the recurrence relations (2.3) let $r = n-m$. Then

$$b_{n-m+1} = \{m^2(1+k^2)-m^2-\alpha k^2+O(k^4)\}b_{n-m} + \\ + k^2(m+\tfrac{1}{2})(m+1)(n-m)(n+m+\tfrac{1}{2})b_{n-m-1}.$$

Thus, equating to zero the coefficient of k^2, we get

$$d_{n-m+1} = (m^2-\alpha)c_{n-m} + (m+\tfrac{1}{2})(m+1)(n-m)(n+m+\tfrac{1}{2})c_{n-m-1}.$$

that is, since $(2m+1)c_{n-m-1} = c_{n-m}$,

$$d_{n-m+1} = \{m^2-\alpha+\tfrac{1}{2}(m+1)(n-m)(n+m+\tfrac{1}{2})\}c_{n-m}. \qquad (2.5)$$

Now let $r = n-m+1$ in the recurrence relations (2.3) and we have

$$b_{n-m+2} = \{(m-1)^2(1+k^2)-m^2-\alpha k^2+O(k^4)\}b_{n-m+1}+ \\ +k^2(m-\tfrac{1}{2})m(n-m+1)(n+m-\tfrac{1}{2})b_{n-m}.$$

It will be seen in the next section that $d_{n-m+2} = 0$. Assuming this for the moment and equating to zero the coefficient of k^2 we have

$$0 = (1-2m)d_{n-m+1}+m(m-\tfrac{1}{2})(n-m+1)(n+m-\tfrac{1}{2})c_{n-m}. \qquad (2.6)$$

From (2.5) and (2.6) we have

$$2(m^2-\alpha)+(m+1)(n-m)(n+m+\tfrac{1}{2}) = m(n+m-\tfrac{1}{2})(n-m+1),$$

$$4\alpha = -2m^2+n(2n+1).$$

Hence $$h = a_{2n}^{2m} = 4m^2-\tfrac{1}{2}k^2\{4m^2-2n(2n+1)\}+O(k^4). \qquad (2.7)$$

The above method applied to each of the different types of Lamé function for n even and n odd leads to a similar result, except in the case $m = 1$, which must be considered separately.

Thus for $\mathscr{E}c_n^m(\rho)$ and $\mathscr{E}s_n^m(\rho)$ $(m \neq 1)$,

$$a_n^m = b_n^m = m^2-\tfrac{1}{2}k^2\{m^2-n(n+1)\}+O(k^4). \qquad (2.8)$$

3. Solutions corresponding to approximate values of h

Substituting the approximate value of h in (1.3) we have

$$(\rho^2-1)(\rho^2-k^2)\frac{d^2u}{d\rho^2}+\rho(2\rho^2-1-k^2)\frac{du}{d\rho}- \\ -\{n(n+1)\rho^2-m^2+\tfrac{1}{2}k^2m^2-\tfrac{1}{2}k^2n(n+1)\}u = 0.$$

Writing $u(\rho) = v(\rho)+k^2w(\rho)$ and equating to zero the coefficient of k^2 we have

$$\rho^2(\rho^2-1)\frac{d^2w}{d\rho^2}+\rho(2\rho^2-1)\frac{dw}{d\rho}-\{n(n+1)\rho^2-m^2\}w \\ = (\rho^2-1)\frac{d^2v}{d\rho^2}+\rho\frac{dv}{d\rho}+\tfrac{1}{2}\{m^2-n(n+1)\}v. \qquad (3.1)$$

Now $$v(\rho) = P_n^m\{\sqrt{(1-\rho^2)}\},$$

and the lowest power of ρ in $v(\rho)$ is ρ^m. Therefore the lowest power of ρ in the expression on the right-hand side of (3.1) is ρ^{m-2}. By inspection of the left-hand side it is easily seen that the lowest power of ρ in $w(\rho)$ cannot be less than ρ^{m-2}. This gives the reason for the assumption in § 2 that $d_{n-m+2} = 0$.

Writing $t^2 = 1-\rho^2$ in (3.1), we obtain

$$(t^2-1)^2\frac{d^2w}{dt^2}+2t(t^2-1)\frac{dw}{dt}-\{n(n+1)(t^2-1)+m^2\}w+$$
$$+(t^2-1)\frac{d^2v}{dt^2}+t\frac{dv}{dt}+\tfrac{1}{2}\{m^2-n(n+1)\}v = 0. \quad (3.2)$$

We can verify by substitution that $w = \frac{1}{4}t(dv/dt)$ is a solution of this equation. Substituting for w we get on the left

$$\frac{1}{4}\Big[t(t^2-1)^2\frac{d^3v}{dt^3}+(4t^2-2)(t^2-1)\frac{d^2v}{dt^2}+\{(2-n^2-n)t(t^2-1)-m^2t\}\frac{dv}{dt}\Big]+$$
$$+(t^2-1)\frac{d^2v}{dt^2}+t\frac{dv}{dt}+\tfrac{1}{2}\{m^2-n(n+1)\}v$$
$$= \frac{1}{4}\Big(t\frac{d}{dt}-2\Big)\Big\{(t^2-1)^2\frac{d^2v}{dt^2}+2t(t^2-1)\frac{dv}{dt}-n(n+1)(t^2-1)v-m^2v\Big\} = 0$$

since $v = P_n^m(t)$.

Therefore the complete solution of (3.2) is

$$w = \tfrac{1}{4}t\frac{d}{dt}P_n^m(t)+AP_n^m(t)+BQ_n^m(t),$$

where A and B are constants. It is evident that $B = 0$, and, since the degree of w is $n-2$ in ρ or t, $A = -\frac{1}{4}n$.

Therefore
$$w = \frac{1}{4}\Big(t\frac{d}{dt}-n\Big)P_n^m(t).$$

Hence we have, for $m \neq 1$,

$$\mathscr{E}c_n^m(\rho) = P_n^m(t)+\tfrac{1}{4}k^2\Big(t\frac{d}{dt}-n\Big)P_n^m(t)+O(k^4),$$

$$\mathscr{E}s_n^m(\rho) = P_n^m(t)+\tfrac{1}{4}k^2\Big(t\frac{d}{dt}-n\Big)P_n^m(t)+O(k^4).$$

4. Solutions when $m = 1$

Let a solution of (2.1) be

$$u = (\rho^2-1)^{\frac{1}{2}}(\rho^2-k^2)^{\frac{1}{2}}\sum_{r=0}^{n-1}\frac{\Gamma(\frac{1}{2}-2n)}{\Gamma(\frac{1}{2}-2n-r)r!}b_r\rho^{2n-2-2r}. \quad (4.1)$$

The recurrence relation obtained by substitution in the differential equation is

$$b_{r+1} = \{(n-\tfrac{1}{2}-r)^2(1+k^2)-\tfrac{1}{4}h\}b_r+k^2r(2n+\tfrac{1}{2}-r)(n-r)(n-\tfrac{1}{2}-r)b_{r-1}. \quad (4.2)$$

The characteristic values of h when $k = 0$ are

$$h = (2m-1)^2 \quad (m = 1, 2, \ldots, n).$$

When $m = 1$, let $h = 1+4\alpha k^2+O(k^4)$, $b_r = c_r+k^2d_r+O(k^4)$.

Then
$$c_r = \frac{(n-1)!\,n!}{(n-1-r)!\,(n-r)!}\,b_0. \tag{4.3}$$

In (4.2) let $r = n-1$; then, since $b_n = 0$, we have

$$0 = \{\tfrac{1}{4}(1+k^2)-\tfrac{1}{4}-\alpha k^2+O(k^4)\}b_{n-1}+\tfrac{1}{2}(n-1)(n+\tfrac{3}{2})k^2 b_{n-2}.$$

Equating to zero the coefficient of k^2 we have

$$(\tfrac{1}{4}-\alpha)c_{n-1}+\tfrac{1}{2}(n-1)(n+\tfrac{3}{2})c_{n-2} = 0,$$

and hence

$$\begin{aligned} 4\alpha &= 1+(n-1)(n+\tfrac{3}{2}) \\ &= -\tfrac{1}{2}\{1-2n(2n+1)\}-\tfrac{1}{4}\,2n(2n+1), \\ h &= 1-\tfrac{1}{2}k^2\{1-2n(2n+1)\}-\tfrac{1}{4}k^2\,2n(2n+1)+O(k^4). \end{aligned}$$

This is the value a^1_{2n} corresponding to $\mathcal{E}c^1_{2n}(\rho)$.

Similarly, for the solution

$$u = \rho(\rho^2-1)^{\frac{1}{2}}\sum_{r=0}^{n-1}\frac{\Gamma(\frac{1}{2}-2n)}{\Gamma(\frac{1}{2}-2n-r)r!}\,b_r\rho^{2n-2-2r},$$

the recurrence relation is

$$b_{r+1} = \{(n-r)^2k^2+(n-\tfrac{1}{2}-r)^2-\tfrac{1}{4}h\}b_r+k^2r(2n+\tfrac{1}{2}-r)(n-r)(n+\tfrac{1}{2}-r)b_{r-1}.$$

Putting $h = 1+4\alpha k^2+O(k^4)$ and $r = n-1$ as before, we have

$$0 = \{k^2(1+\alpha)+O(k^4)\}b_{n-1}+\tfrac{3}{2}k^2(n-1)(n+\tfrac{3}{2})b_{n-2},$$

and hence, using (4.3), we have

$$\begin{aligned} 4\alpha &= -\tfrac{1}{2}\{1-2n(2n+1)\}+\tfrac{1}{4}\,2n(2n+1), \\ h &= 1-\tfrac{1}{2}k^2\{1-2n(2n+1)\}+\tfrac{1}{4}k^2\,2n(2n+1)+O(k^4). \end{aligned}$$

This is the value b^1_{2n} corresponding to $\mathcal{E}s^1_{2n}(\rho)$.

Similar results are obtained for Lamé functions of odd order, and we have in general

$$\begin{aligned} a^1_n &= 1-\tfrac{1}{2}k^2\{1-n(n+1)\}-\tfrac{1}{4}k^2n(n+1)+O(k^4), \\ b^1_n &= 1-\tfrac{1}{2}k^2\{1-n(n+1)\}+\tfrac{1}{4}k^2n(n+1)+O(k^4). \end{aligned}$$

In the case when $m = 1$ the equation (3.2) becomes

$$(t^2-1)^2\frac{d^2w}{dt^2}+2t(t^2-1)\frac{dw}{dt}-\{n(n+1)(t^2-1)+1\}w+$$

$$+(t^2-1)\frac{d^2v}{dt^2}+t\frac{dv}{dt}+\tfrac{1}{2}v-\tfrac{1}{2}n(n+1)v\pm\tfrac{1}{4}n(n+1)v = 0.$$

A solution is

$$w = (\tfrac{1}{4}\pm\tfrac{1}{8})\left(t\frac{d}{dt}-n\right)P^1_n(t)\mp\tfrac{1}{8}(1-t^2)^{-1}P^1_n(t),$$

and hence

$$\mathscr{E}c_n^1(\rho) = P_n^1(t)+\tfrac{3}{8}k^2\left(t\frac{d}{dt}-n\right)P_n^1(t)-\tfrac{1}{8}k^2(1-t^2)^{-1}P_n^1(t)+O(k^4),$$

$$\mathscr{E}s_n^1(\rho) = P_n^1(t)+\tfrac{1}{8}k^2\left(t\frac{d}{dt}-n\right)P_n^1(t)+\tfrac{1}{8}k^2(1-t^2)^{-1}P_n^1(t)+O(k^4).$$

5. Functions of the second kind

Substituting in (1.5) the approximate value obtained for $\mathscr{E}c_n^m(\rho)$, we have, when $m \neq 1$,

$$\mathscr{E}c_n^m\{\sqrt{(1-s^2)}\} = P_n^m(s)+\tfrac{1}{4}k^2\left(s\frac{d}{ds}-n\right)P_n^m(s)+O(k^4),$$

$$\begin{aligned}\mathscr{F}c_n^m(\rho) &= Q_n^m(t)+\tfrac{1}{4}k^2\{P_n^m(t)\}^{-1}Q_n^m(t)\left(t\frac{d}{dt}-n\right)P_n^m(t)+\\ &\quad+\frac{(n+m)!}{2(n-m)!}P_n^m(t)\int_t^\infty\frac{[P_n^m(s)]^{-2}}{s^2-1}\times\\ &\qquad\times\left[-\frac{k^2}{s^2-1}-k^2\{P_n^m(s)\}^{-1}s\frac{d}{ds}P_n^m(s)+nk^2\right]ds\\ &= Q_n^m(t)+\tfrac{1}{4}k^2\{P_n^m(t)\}^{-1}Q_n^m(t)t\frac{d}{dt}P_n^m(t)+n\tfrac{1}{4}k^2Q_n^m(t)+\\ &\quad+\frac{k^2(n+m)!}{4(n-m)!}P_n^m(t)\int_t^\infty\left[\frac{d}{ds}\left\{\frac{s[P_n^m(s)]^{-2}}{s^2-1}\right\}+\frac{[P_n^m(s)]^{-2}}{s^2-1}\right]ds\\ &= Q_n^m(t)+\tfrac{1}{4}k^2(n+1)Q_n^m(t)+\tfrac{1}{4}k^2[P_n^m(t)]^{-1}Q_n^m(t)t\frac{d}{dt}P_n^m(t)-\\ &\qquad-\frac{k^2(n+m)!}{4(n-m)!}\frac{t}{(t^2-1)}\{P_n^m(t)\}^{-1}.\end{aligned}$$

Now $$Q_n^m(t)\frac{d}{dt}P_n^m(t)-P_n^m(t)\frac{d}{dt}Q_n^m(t) = \frac{(n+m)!}{(n-m)!\,(t^2-1)}.$$

Therefore, when $m \neq 1$,

$$\mathscr{F}c_n^m(\rho) = \mathscr{F}s_n^m(\rho) = Q_n^m(t)+\tfrac{1}{4}k^2\left(t\frac{d}{dt}+n+1\right)Q_n^m(t)+O(k^4).$$

Similarly, when $m = 1$, we find

$$\mathscr{F}c_n^1(\rho) = Q_n^1(t)+\tfrac{3}{8}k^2\left(t\frac{d}{dt}+n+1\right)Q_n^1(t)+\tfrac{1}{8}k^2(t^2-1)^{-1}Q_n^1(t)+O(k^4),$$

$$\mathscr{F}s_n^1(\rho) = Q_n^1(t)+\tfrac{1}{8}k^2\left(t\frac{d}{dt}+n+1\right)Q_n^1(t)-\tfrac{1}{8}k^2(t^2-1)^{-1}Q_n^1(t)+O(k^4).$$

6. Lamé products

Expressions for Lamé products follow from the previous section. We have, when $m \neq 1$,

$$\mathscr{E}c_n^m(\rho)\mathscr{F}c_n^m(\rho) = P_n^m(t)Q_n^m(t)+\tfrac{1}{4}k^2\frac{d}{dt}tP_n^m(t)Q_n^m(t)+O(k^4),$$

$$\mathscr{E}s_n^m(\rho)\mathscr{F}s_n^m(\rho) = P_n^m(t)Q_n^m(t)+\tfrac{1}{4}k^2\frac{d}{dt}tP_n^m(t)Q_n^m(t)+O(k^4).$$

Also

$$\mathscr{E}c_n^1(\rho)\mathscr{F}c_n^1(\rho)$$
$$= P_n^1(t)Q_n^1(t)+\tfrac{3}{8}k^2\frac{d}{dt}tP_n^1(t)Q_n^1(t)-\tfrac{1}{4}k^2(t^2-1)^{-1}P_n^1(t)Q_n^1(t)+O(k^4),$$

$$\mathscr{E}s_n^1(\rho)\mathscr{F}s_n^1(\rho)$$
$$= P_n^1(t)Q_n^1(t)+\tfrac{1}{8}k^2\frac{d}{dt}tP_n^1(t)Q_n^1(t)+\tfrac{1}{4}k^2(t^2-1)^{-1}P_n^1(t)Q_n^1(t)+O(k^4).$$

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GENERALIZED MODIFIED STRUVE FUNCTIONS

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1. Introduction

In this paper we consider two generalized modified Struve functions $\Omega(a, c; x)$ and $\overline{\Omega}(a, c; x)$, defined in §§ 3 and 5. As shown in § 7 these functions are particular integrals of the inhomogeneous confluent hypergeometric equations

$$x\frac{d^2y}{dx^2}+(c-x)\frac{dy}{dx}-ay=\frac{2^{1-c}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}e^{\frac{1}{2}x}, \tag{1}$$

$$x\frac{d^2y}{dx^2}+(c-x)\frac{dy}{dx}-ay=\frac{2^{c-1}\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(1-a)}e^{\frac{1}{2}x}x^{1-c}. \tag{2}$$

2. Integral relations for confluent hypergeometric functions

For ease of reference, we first summarize various integral relations for the solutions of the equation

$$x\frac{d^2y}{dx^2}+(c-x)\frac{dy}{dx}-ay=0, \tag{3}$$

where a and c are constants.

From (**1**) four such solutions are

$$\Phi(a, c; x) = {}_1F_1(a, c; x), \tag{4}$$

$$\overline{\Phi}(a, c; x) = x^{1-c}\Phi(a-c+1, 2-c; x), \tag{5}$$

$$\Psi(a, c; x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a, c; x)+\frac{\Gamma(c-1)}{\Gamma(a)}\overline{\Phi}(a, c; x), \tag{6}$$

$$\overline{\Psi}(a, c; x) = e^x\Psi(c-a, c; -x). \tag{7}$$

When $c = 2a$, as shown in [(**1**) 265] the above solutions can be expressed in terms of modified Bessel functions. Thus

$$\Phi(a, 2a; x) = \Gamma(a+\tfrac{1}{2})(\tfrac{1}{4}x)^{\frac{1}{2}-a}e^{\frac{1}{2}x}I_{a-\frac{1}{2}}(\tfrac{1}{2}x), \tag{8}$$

$$\overline{\Phi}(a, 2a; x) = \Gamma(\tfrac{3}{2}-a)(4x)^{\frac{1}{2}-a}e^{\frac{1}{2}x}I_{\frac{1}{2}-a}(\tfrac{1}{2}x). \tag{9}$$

As shown in [(3) 143] the solutions of (3) can be expressed in terms of the contour integrals

$$y_1 = \frac{1}{2\pi i}\int_I^{(1+)} e^{xt}t^{a-1}(t-1)^{c-a-1}\,dt, \tag{10}$$

$$y_2 = \frac{1}{2\pi i}\int_I^{(0+)} e^{xt}t^{a-1}(t-1)^{c-a-1}\,dt, \tag{11}$$

where I is the point at infinity taken in such a direction that xt is real and negative. The contours for these integrals are shown in Fig. 1.

Thus, if $\qquad \arg x = \theta \quad (-\pi < \theta < \pi),$

we take I as the point at infinity having argument $\pi-\theta$ and

$$\arg t = \arg(t-1) = -\pi-\theta$$

at the start of the contour.

From (1) and (3),

$$\Phi(a,c;x) = i(2\pi)^{-1}e^{i\pi a}\Gamma(1-a)\Gamma(c)\Gamma(1+a-c)y_3, \tag{12}$$

$$\overline{\Phi}(a,c;x) = \Gamma(2-c)y_4, \tag{13}$$

$$\Psi(a,c;x) = -e^{i\pi(c-a)}\Gamma(1-a)y_2, \tag{14}$$

$$\overline{\Psi}(a,c;x) = e^{i\pi(a+c)}\Gamma(1+a-c)y_1 \quad (\operatorname{im} x > 0), \tag{15}$$

$$\overline{\Psi}(a,c;x) = e^{i\pi(a-c)}\Gamma(1+a-c)y_5 \quad (\operatorname{im} x < 0), \tag{16}$$

where

$$\left.\begin{aligned} y_3 &= \frac{1}{2\pi i}\int_I^{(1+,0+,1-,0-)} e^{xt}t^{a-1}(t-1)^{c-a-1}\,dt \\ &= y_1[1-e^{2\pi ia}]-y_2[1-e^{2\pi i(c-a)}] \end{aligned}\right\}, \tag{17}$$

$$\left.\begin{aligned} y_4 &= \frac{1}{2\pi i}\int_I^{(0+,1+)} e^{xt}t^{a-1}(t-1)^{c-a-1}\,dt \\ &= y_2+y_1e^{2\pi ia} \end{aligned}\right\}, \tag{18}$$

$$y_5 = y_1e^{2\pi ia}+y_2[1-e^{2\pi i(c-a)}]. \tag{19}$$

The contour for y_5 is shown in Fig. 1.

As shown in (1), the path of integration for y_3 may be contracted to a finite contour $(1+,0+,1-,0-)$. Finally, putting $t = \frac{1}{2}(u+1)$, we

find that (17) can be reduced to

$$y_3 = i\frac{2-c}{\pi}e^{-i\pi(a+c)}e^{\frac{1}{2}x}\left[(1-e^{2i\pi a})\int_0^{(1+)} e^{\frac{1}{2}xu}(1+u)^{a-1}(1-u)^{c-a-1}\,du - \right.$$
$$\left. -\{1-e^{2i\pi(c-a)}\}\int_0^{(-1+)} e^{\frac{1}{2}xu}(1+u)^{a-1}(1-u)^{c-a-1}\,du\right], \quad (20)$$

where the arguments of $1+u$ and $1-u$ are both to be taken as zero at the start of the contours.

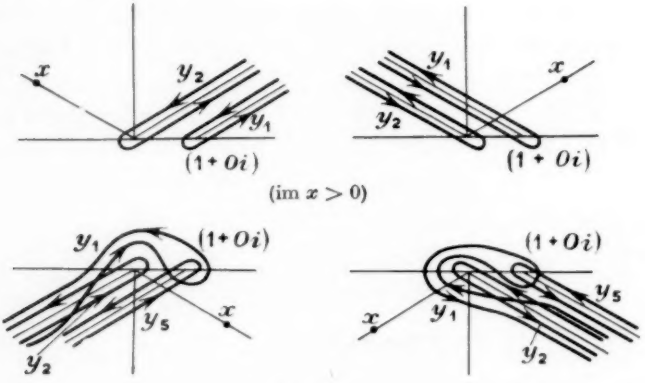

FIG. 1 Contours for y_1, y_2, y_3.

For $\mathrm{re}\,c > \mathrm{re}\,a > 0$, the contours of integration in (20) may be replaced by portions of the real axis. We find that

$$\Phi(a,c;x) = 2^{2-c}\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}e^{\frac{1}{2}x}\int_0^1 (1-u^2)^{\frac{1}{2}c-1}\times$$
$$\times\cosh\left[\tfrac{1}{2}xu+(a-\tfrac{1}{2}c)\log\frac{1+u}{1-u}\right]du. \quad (21)$$

Put $u = \cos\theta$. Then (21) becomes

$$\Phi(a,c;x) = 2^{2-c}\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}e^{\frac{1}{2}x}\int_0^{\frac{1}{2}\pi} (\sin\theta)^{c-1}\times$$
$$\times\cosh[\tfrac{1}{2}x\cos\theta-(2a-c)\log\tan\tfrac{1}{2}\theta]\,d\theta. \quad (22)$$

From (8), the integral formulae for $I_\nu(z)$ can be obtained immediately by putting $c = 2a$ in (21) and (22).

3. Generalized modified Struve function $\Omega(a,c;x)$

Consider the function $\Omega(a,c;x)$ defined by

$$\Omega(a,c;x) = i(2\pi)^{-1}e^{i\pi a}\Gamma(1-a)\Gamma(c)\Gamma(1+a-c)\omega, \tag{23}$$

where

$$\omega = i\frac{2-c}{\pi}e^{-i\pi(a+c)}e^{\frac{1}{2}x}\Big[(1-e^{2i\pi a})\int_0^{(1+)} e^{\frac{1}{2}xu}(1+u)^{a-1}(1-u)^{c-a-1}\,du + \\ +\{1-e^{2i\pi(c-a)}\}\int_0^{(-1+)} e^{\frac{1}{2}xu}(1+u)^{a-1}(1-u)^{c-a-1}\,du\Big]. \tag{24}$$

From (12) and (23), $$\frac{\Omega(a,c;x)}{\Phi(a,c;x)} = \frac{\omega}{y_3}.$$

In (24), the arguments of $1+u$ and $1-u$ are both to be taken as zero at the start of the contours.

For $\operatorname{re} c > \operatorname{re} a > 0$, the contours of integration in (24) may be replaced by portions of the real axis. We find that

$$\Omega(a,c;x) = 2^{2-c}\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}e^{\frac{1}{2}x}\int_0^1 (1-u^2)^{\frac{1}{2}c-1}\times \\ \times\sinh\Big[\tfrac{1}{2}xu+(a-\tfrac{1}{2}c)\log\frac{1+u}{1-u}\Big]\,du. \tag{25}$$

Put $u = \cos\theta$. Then (25) becomes

$$\Omega(a,c;x) = 2^{2-c}\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}e^{\frac{1}{2}x}\int_0^{\frac{1}{2}\pi} (\sin\theta)^{c-1}\times \\ \times\sinh[\tfrac{1}{2}x\cos\theta-(2a-c)\log\tan\tfrac{1}{2}\theta]\,d\theta. \tag{26}$$

When $c = 2a$, from [(**2**) 38],

$$\Omega(a,2a;x) = \Gamma(a+\tfrac{1}{2})(\tfrac{1}{4}x)^{\frac{1}{2}-a}e^{\frac{1}{2}x}L_{a-\frac{1}{2}}(\tfrac{1}{2}x), \tag{27}$$

where $L_{a-\frac{1}{2}}(\frac{1}{2}x)$ is the modified Struve function.

From (24), putting $t = \frac{1}{2}(u+1)$, we see that

$$\omega = \frac{1}{2\pi i}\Big[(1-e^{2\pi ia})\int_1^{(1+)} e^{xt}t^{a-1}(t-1)^{c-a-1}\,dt + \\ +\{1-e^{2\pi i(c-a)}\}\int_1^{(0+)} e^{xt}t^{a-1}(t-1)^{c-a-1}\,dt\Big], \tag{28}$$

where $\arg t = -2\pi$ and $\arg(t-1) = -\pi$ at the start of the contour.

Using (10), (11), and (19) we see, by deforming the contours, that the formula (28) can be put in the form

$$\omega = \frac{1}{2\pi i}\Big[(1-e^{2\pi i a})\Big(\int_{\frac{1}{2}}^{I} X\,dt + 2\pi i y_1 - e^{2\pi i(c-a)}\int_{\frac{1}{2}}^{I} X\,dt\Big) + \\ + \{1-e^{2\pi i(c-a)}\}\Big(\int_{\frac{1}{2}}^{I} X\,dt + 2\pi i y_2 - e^{2\pi i a}\int_{\frac{1}{2}}^{I} X\,dt\Big)\Big] \quad (\mathrm{im}\,x > 0), \quad (29)$$

$$\omega = \frac{1}{2\pi i}\Big[(1-e^{2\pi i a})\Big(\int_{\frac{1}{2}}^{I} X\,dt + 2\pi i e^{-2\pi i a} y_5 - e^{2\pi i(c-a)}\int_{\frac{1}{2}}^{I} X\,dt\Big) + \\ + \{1-e^{2\pi i(c-a)}\}\Big(\int_{\frac{1}{2}}^{I} X\,dt + 2\pi i e^{-2\pi i a} y_2 - e^{2\pi i a}\int_{\frac{1}{2}}^{I} X\,dt\Big)\Big] \quad (\mathrm{im}\,x < 0), \quad (30)$$

where $$X = e^{xt}t^{a-1}(t-1)^{c-a-1},$$

and the contours for y_1, y_2, y_5 are given above.

The integrals $\int_{\frac{1}{2}}^{I} X\,dt$ can be taken along the line through $t = \frac{1}{2}$ having argument $\pi-\theta$, where $\arg x = \theta$ $(-\pi < \theta < \pi)$.

Putting $t = \frac{1}{2}+v$, we find that

$$\int_{\frac{1}{2}}^{I} X\,dt = e^{-2\pi i c}e^{\frac{1}{2}x}\int_{0}^{\infty\exp(i\pi-i\theta)} e^{xv}(v+\tfrac{1}{2})^{a-1}(v-\tfrac{1}{2})^{c-a-1}\,dv, \quad (31)$$

where the arguments of $v+\frac{1}{2}$ and $v-\frac{1}{2}$ are to be taken as 0 and π at the origin.

From (23), using (14)–(16), we find that

$$\Omega(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)}e^{i\pi(a-c)\epsilon}e^{x}\Psi(c-a,c;-x) - \frac{\Gamma(c)}{\Gamma(c-a)}e^{i\pi a\epsilon}\Psi(a,c;x) - \\ - \frac{2e^{i\pi(a-c)}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}e^{\frac{1}{2}x}\int_{0}^{\infty\exp(i\pi-i\theta)} e^{xv}(v+\tfrac{1}{2})^{a-1}(v-\tfrac{1}{2})^{c-a-1}\,dv, \quad (32)$$

where $\epsilon = \mathrm{sgn}(\mathrm{im}\,x) = 1, -1$ according as $\mathrm{im}\,x > 0, < 0$.

4. An alternative integral relation for $\Omega(a,c;x)$

When x is real and positive and $\mathrm{re}\,c < 1$, the contours for the two integrals in (28) can be deformed as shown in Fig. 2, part of each contour being a semicircle of large radius. Then, for $x > 0$, $\mathrm{re}\,c < 1$, the

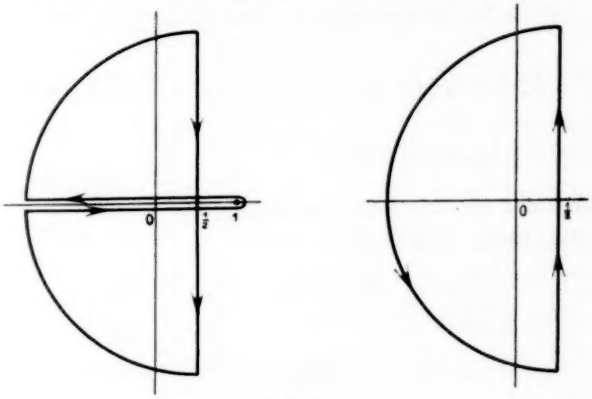

FIG. 2

contribution from the semicircles vanishes as their radius increases to infinity and we find that

$$\omega = \frac{1}{2\pi i}\Big[(1-e^{2\pi i a})\Big\{\int_{\frac{1}{2}}^{\frac{1}{2}-\infty i} X\,dt + 2\pi i e^{-2\pi i a} y_4 - e^{2\pi i(c-a)}\int_{\frac{1}{2}}^{\frac{1}{2}+\infty i} X\,dt\Big\} +$$

$$+\{1-e^{2\pi i(c-a)}\}\Big\{\int_{\frac{1}{2}}^{\frac{1}{2}+\infty i} X\,dt - e^{2\pi i a}\int_{\frac{1}{2}}^{\frac{1}{2}-\infty i} X\,dt\Big\}\Big],$$

i.e.
$$\omega = (e^{-2\pi i a}-1)y_4 + \frac{1}{2\pi i}[1-2e^{2\pi i a}+e^{2\pi i c}]\int_{\frac{1}{2}}^{\frac{1}{2}-\infty i} X\,dt +$$

$$+\frac{1}{2\pi i}[1-2e^{2\pi i(c-a)}+e^{2\pi i c}]\int_{\frac{1}{2}}^{\frac{1}{2}+\infty i} X\,dt. \quad (33)$$

Equation (33) can be simplified to give

$$\omega = (e^{-2\pi i a}-1)y_4 - i\pi^{-1}e^{-i\pi a}e^{\frac{1}{2}x}[\mathrm{re}\, Z(a,c;x)], \quad (34)$$

where

$$Z(a,c;x) = ie^{-i\pi(a-\frac{1}{2}c)}\{e^{2\pi i a}-2+e^{-2\pi i(c-a)}\}\times$$

$$\times\int_0^\infty e^{isx}(s-\tfrac{1}{2}i)^{a-1}(s+\tfrac{1}{2}i)^{c-a-1}\,ds. \quad (35)$$

From (13), (23), (34), if $x > 0$ and $\mathrm{re}\, c < 1$,

$$\Omega(a,c;x) = \frac{\Gamma(c)\Gamma(1+a-c)}{\Gamma(2-c)\Gamma(a)}\overline{\Phi}(a,c;x) +$$

$$+(2\pi^2)^{-1}\Gamma(1-a)\Gamma(c)\Gamma(1+a-c)e^{\frac{1}{2}x}[\mathrm{re}\, Z(a,c;x)]. \quad (36)$$

When $c = 2a$, using (9) and (27), we obtain

$$L_{a-\frac{1}{2}}(\tfrac{1}{2}x) = I_{\frac{1}{2}-a}(\tfrac{1}{2}x) - \frac{2(\frac{1}{4}x)^{a-\frac{1}{2}}}{\sqrt{\pi}\Gamma(a)} \int_0^\infty (1+u^2)^{a-1} \sin \tfrac{1}{2}xu \, du, \tag{37}$$

in agreement with (**2**) [38 (54)].

5. Generalized modified Struve function $\overline{\Omega}(a, c; x)$

By analogy with (5) we define

$$\overline{\Omega}(a, c; x) = x^{1-c}\Omega(a-c+1; 2-c; x). \tag{38}$$

When $c = 2a$, from (27),

$$\overline{\Omega}(a, 2a; x) = \Gamma(\tfrac{3}{2}-a)(4x)^{\frac{1}{2}-a}e^{\frac{1}{2}x}L_{\frac{1}{2}-a}(\tfrac{1}{2}x). \tag{39}$$

From (32), (38), using the relations for Ψ given in [(**1**) 257], we see that

$$\begin{aligned}\overline{\Omega}(a, c; x) &= \frac{\Gamma(2-c)}{\Gamma(1-a)} e^{i\pi(a-c)\epsilon}\Psi(a, c; x) - \frac{\Gamma(2-c)}{\Gamma(a-c+1)} e^{i\pi a\epsilon}e^{x}\Psi(c-a, c; -x) + \\ &+ \frac{2e^{\pi i a}\Gamma(2-c)x^{1-c}}{\Gamma(1-a)\Gamma(a-c+1)} e^{\frac{1}{2}x} \int_0^{\infty \exp(i\pi - i\theta)} e^{xv}(v+\tfrac{1}{2})^{a-c}(v-\tfrac{1}{2})^{-a} \, dv.\end{aligned} \tag{40}$$

We note that equations (32) and (40) are true for all values of a and c.

6. Relations between contiguous functions

Consider the integral

$$I(a, c; x) = \int_0^{\infty \exp(i\pi - i\theta)} e^{x(v+\frac{1}{2})}(v+\tfrac{1}{2})^{a-1}(v-\tfrac{1}{2})^{c-a-1} \, dv. \tag{41}$$

Now
$$I(a+1, c; x) = I(a, c; x) + I(a, c-1; x). \tag{42}$$

Integrating (41) by parts we obtain

$$xI(a, c; x) = e^{i\pi(c-a)}2^{2-c}e^{\frac{1}{2}x} - (a-1)I(a-1, c-1; x) - (c-a-1)I(a, c-1; x),$$

i.e.

$$(c-a)I(a, c; x) + (a-1)I(a-1, c; x) + xI(a, c+1; x) = -e^{i\pi(c-a)}2^{1-c}e^{\frac{1}{2}x}. \tag{43}$$

From (41),
$$\frac{dI(a, c; x)}{dx} = I(a+1, c+1; x). \tag{44}$$

From (32), (42), (43), using the recurrence formulae for Ψ given in (1), we find that

$$(a-c+1)\Omega(a,c;x)-a\Omega(a+1,c;x)+(c-1)\Omega(a,c-1;x)=0, \quad (45)$$

$$\Omega(a,c;x)-\Omega(a-1,c;x)-xc^{-1}\Omega(a,c+1;x)=\frac{2^{2-c}\Gamma(c)}{\Gamma(a)\Gamma(c-a+1)}e^{\frac{1}{2}x}. \quad (46)$$

From (45) and (46),

$$(a+x)\Omega(a,c;x)-\frac{(c-a)x}{c}\Omega(a,c+1;x)-a\Omega(a+1,c;x) = -\frac{2^{2-c}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}e^{\frac{1}{2}x}. \quad (47)$$

From (46) and (47),

$$(c-a)\Omega(a-1,c;x)+(2a-c+x)\Omega(a,c;x)-a\Omega(a+1,c;x) = -\frac{2^{3-c}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}e^{\frac{1}{2}x}. \quad (48)$$

From (45) and (47),

$$(c-1)\Omega(a,c-1;x)-(c-1+x)\Omega(a,c;x)+\frac{(c-a)x}{c}\Omega(a,c+1;x) = \frac{2^{2-c}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}e^{\frac{1}{2}x}. \quad (49)$$

From (45) and (48),

$$(a-1+x)\Omega(a,c;x)+(c-a)\Omega(a-1,c;x)-(c-1)\Omega(a,c-1;x) = -\frac{2^{3-c}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}e^{\frac{1}{2}x}. \quad (50)$$

From (45), (46), (48),

$$(c-1)\Omega(a-1,c-1;x)+(1-c+x)\Omega(a,c;x)-\frac{ax}{c}\Omega(a+1,c+1;x) = -\frac{2^{2-c}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}e^{\frac{1}{2}x}. \quad (51)$$

Similarly, using (38), we find that

$$\overline{\Omega}(a,c;x)-\overline{\Omega}(a+1,c;x)-\frac{1}{c-2}\overline{\Omega}(a,c-1;x) = -\frac{2^{c}\Gamma(2-c)}{\Gamma(a-c+2)\Gamma(1-a)}e^{\frac{1}{2}x}x^{1-c}, \quad (52)$$

$$(c-a)\overline{\Omega}(a,c;x)+(a-1)\overline{\Omega}(a-1,c;x)-(c-1)x\overline{\Omega}(a,c+1;x)=0, \quad (53)$$

$$(a+x)\overline{\Omega}(a,c;x)-(c-1)x\overline{\Omega}(a,c+1;x)+(c-a-1)\overline{\Omega}(a+1,c;x) = -\frac{2^{c+1}\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(1-a)}e^{\frac{1}{2}x}x^{1-c}, \quad (54)$$

$$(a-1)\overline{\Omega}(a-1,c;x)-(2a-c+x)\overline{\Omega}(a,c;x)-(c-a-1)\overline{\Omega}(a+1,c;x) = \frac{2^{c+1}\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(1-a)}e^{\frac{1}{2}x}x^{1-c}, \quad (55)$$

$$\frac{c-a-1}{c-2}\overline{\Omega}(a,c-1;x)-(c-1+x)\overline{\Omega}(a,c;x)+(c-1)x\overline{\Omega}(a,c+1;x) = \frac{2^{c}\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(1-a)}e^{\frac{1}{2}x}x^{1-c}, \quad (56)$$

$$(a-1+x)\overline{\Omega}(a,c;x)-(a-1)\overline{\Omega}(a-1,c;x)-\frac{c-a-1}{c-2}\overline{\Omega}(a,c-1;x) = -\frac{2^{c}\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(1-a)}e^{\frac{1}{2}x}x^{1-c}, \quad (57)$$

$$\frac{a-1}{c-2}\overline{\Omega}(a-1,c-1;x)-(1-c+x)\overline{\Omega}(a,c;x)-(c-1)x\overline{\Omega}(a+1,c+1;x) = \frac{2^{c}\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(1-a)}e^{\frac{1}{2}x}x^{1-c}. \quad (58)$$

7. Differential equations satisfied by $\Omega(a,c;x)$ and $\overline{\Omega}(a,c;x)$

From (32) and (44), using the recurrence formulae for Ψ given in (1), we see that

$$\frac{d\Omega(a,c;x)}{dx} = \frac{a}{c}\Omega(a+1,c+1;x), \quad (59)$$

$$\frac{d^{n}\Omega(a,c;x)}{dx^{n}} = \frac{\Gamma(a+n)\Gamma(c)}{\Gamma(a)\Gamma(c+n)}\Omega(a+n,c+n;x). \quad (60)$$

From (51), (59), (60) we find that $\Omega(a,c;x)$ satisfies the differential equation

$$x\frac{d^{2}y}{dx^{2}}+(c-x)\frac{dy}{dx}-ay = \frac{2^{1-c}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}e^{\frac{1}{2}x}. \quad (61)$$

Similarly, using (38), we find that

$$\frac{d\overline{\Omega}(a,c;x)}{dx} = -(c-1)\overline{\Omega}(a+1,c+1;x)-\frac{2^{c}\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(1-a)}e^{\frac{1}{2}x}x^{-c}, \quad (62)$$

$$\frac{d^{2}\overline{\Omega}(a,c;x)}{dx^{2}} = c(c-1)\overline{\Omega}(a+2,c+2;x)+ +\frac{2^{c}\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(1-a)}e^{\frac{1}{2}x}x^{-c-1}[2a+c-\tfrac{1}{2}x]. \quad (63)$$

From (58), (62), (63), we find that $\overline{\Omega}(a,c;x)$ satisfies the differential equation

$$x\frac{d^{2}y}{dx^{2}}+(c-x)\frac{dy}{dx}-ay = \frac{2^{c-1}\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(1-a)}e^{\frac{1}{2}x}x^{1-c}. \quad (64)$$

From (59) we find that

$$\frac{d^n}{dx^n}[e^{-x}\Omega(a,c;x)] = (-1)^n \frac{\Gamma(c-a+n)\Gamma(c)}{\Gamma(c-a)\Gamma(c+n)} e^{-x}\Omega(a,c+n;x). \quad (65)$$

8. Generalized modified Struve function $A_{\kappa,\mu}(x)$

From [(1) 264], the Whittaker function $M_{\kappa,\mu}(x)$ is given by

$$M_{\kappa,\mu}(x) = e^{-\frac{1}{2}x}x^{\frac{1}{2}c}\Phi(a,c;x), \quad (66)$$

where

$$\kappa = \tfrac{1}{2}c-a, \qquad \mu = \tfrac{1}{2}c-\tfrac{1}{2}. \quad (67)$$

By analogy with (66) we define the function $A_{\kappa,\mu}(x)$ by the equation

$$A_{\kappa,\mu}(x) = e^{-\frac{1}{2}x}x^{\frac{1}{2}c}\Omega(a,c;x), \quad (68)$$

where κ and μ are given by (67). From (38),

$$A_{\kappa,-\mu}(x) = e^{-\frac{1}{2}x}x^{\frac{1}{2}c}\overline{\Omega}(a,c;x). \quad (69)$$

When $\kappa = 0$, $c = 2a$, from (27),

$$A_{0,\mu}(x) = 2^{2\mu}\Gamma(\mu+1)x^{\frac{1}{2}}L_{\mu}(\tfrac{1}{2}x). \quad (70)$$

From (61) and (68), $A_{\kappa,\mu}(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2}+\left(-\frac{1}{4}+\frac{\kappa}{x}+\frac{\frac{1}{4}-\mu^2}{x^2}\right)y = \frac{2^{-2\mu}\Gamma(2\mu+1)}{\Gamma(\mu-\kappa+\frac{1}{2})\Gamma(\mu+\kappa+\frac{1}{2})}x^{\mu-\frac{1}{2}}. \quad (71)$$

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SOME REMARKS ON BOREL SUMMABILITY

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1. Introduction

THESE remarks are suggested by the analyses of two recent papers (**2**, **4**) whose notation and definitions are adopted here also, as regards the Hölder method (H, α) for any real number α, the Borel exponential method (B), the Hausdorff sequence-to-sequence method (H, μ_n), the quasi-Hausdorff sequence-to-sequence method (H^*, μ_{n+1}) [see also Remark 1 following the proof of Theorem 2 below], the Euler–Knopp method $E(p)$, and the 'circle-method' $T(p)$, each of the last two methods for $0 < p < 1$. The sequences transformed by these methods are supposed to be real unless otherwise stated. In the present note I show that the analogues for the method (B) of the Tauberian theorems for the Abel-method given in (2) hold, without any exceptional case as in (2). In § 3 I give a more complete form of a result due to Ramanujan [(**4**) Theorem 8] on the summability of Borel-summable bounded sequences by Hausdorff and quasi-Hausdorff methods. It may be noted that, whereas Ramanujan's proof of his weaker form of the second part of Theorem 2 of § 3 makes use of Lorentz's theory of 'almost convergent' sequences, the proof given by me here is independent of this theory and is therefore more elementary. Indeed, the theorems of Lorentz and Ramanujan on the strong regularity of Hausdorff and quasi-Hausdorff methods respectively, and of Hill and Lorentz on the Borel property of regular methods are shown to be corollaries of Theorem 2 of § 3.

2. Tauberian theorems

In this section we use the phrase 'summability method' in a general sense: by a *summability method*, say A, we mean any method by which there is associated with every member of some class of sequences ('A-summable sequences') a number ('the A-sum of the sequence'), and which has the properties (i)–(iii) given below:

(i) *A sums all convergent sequences*;

(ii) *if $[s] \equiv \{s_n\}$, $[t] \equiv \{t_n\}$ are summable-A to l_1, l_2 respectively, then $[s+t] \equiv \{s_n+t_n\}$ is summable-A to l_1+l_2*;

(iii) *if $[s] \equiv \{s_n\}$ is summable-A to l and if λ is any constant, then $[\lambda s] \equiv \{\lambda s_n\}$ is summable-A to λl.*

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We consider those summability methods A which satisfy, in addition to (i)–(iii) above, some or all of a number of propositions (I)–(IV) given below. Throughout this section the sequences concerned may be real, or complex.

(I) *If*
$$s_n = \frac{\Gamma(n+1)}{\Gamma(n+1-c)},$$
where $\mathrm{re}\, c > 0$, *then* $\{s_n\}$ *is* not *summable-A.*

(II) *If* $[s] \equiv \{s_n\}$ *is summable-A to l, then the Hölder transform* $H^1[s]$ *is also summable-A to l.*†

(III) *If* $[s] \equiv \{s_n\}$ *is summable-A and if*
$$s_n - s_{n-1} = o(n^{-1}), \tag{1}$$
then $\{s_n\}$ *is convergent.*

(IV) *If* $[s] = \{s_n\}$ *is summable-A and if*
$$s_n = o(\log n), \tag{2}$$
then $\{s_n\}$ *is summable-*(H, β) *for some real number* β.

Proposition (I) is related to total regularity, (II) is a 'product theorem' satisfied by many well-known summability methods, and (III) is a Tauberian proposition which is true for most of the usual summability methods. Also, (1) implies (2) and therefore (IV) implies (III).

THEOREM 1. *Let* $[s] \equiv \{s_n\}$ *be a complex sequence summable-A, where A satisfies* (i), (ii) *and* (iii). *For some real* α *and complex c let*
$$[w] \equiv (H^\alpha - (c+1)H^{\alpha+1})[s] \quad \textit{converge (to } l_0\textit{, say)}. \tag{3}$$
Then the following statements are true.

(*a*) *If* (I), (II) *hold and if* $\mathrm{re}\, c \neq 0$, *then* $H^\alpha[s]$ *is convergent.*

(*b*) *If* (I), (II), (III) *hold, then* $H^\alpha[s]$ *is convergent also when* $c = 0$, *but perhaps not necessarily when* $\mathrm{re}\, c = 0$ $(c \neq 0)$.

(*c*) *If* (I), (II), (IV) *hold, then* $H^\alpha[s]$ *is convergent whatever be c in* (3).

Proof. If $\mathrm{re}\, c < 0$, then by Mercer's theorem [(1) Theorem 51] it follows directly from (3) alone that $H^\alpha[s]$ is convergent. Therefore we let $\mathrm{re}\, c \geqslant 0$ in what follows.

Proof of (*a*). From (3) we see that the integer k can be chosen such that
$$(H^0 - (c+1)H^1)H^k[s] \text{ converges to } l_0. \tag{4}$$

† It is in fact enough if $H^\beta[s]$ is summable-A to l for some real β. We take $\beta = 1$ for simplicity.

Now, it follows from repeated applications of (II) that $H^k[s]$ is A-summable; (I) shows then that (in the case $\operatorname{re} c > 0$)

$$H^k[s] \text{ is convergent.} \tag{5}$$

We can therefore find an integer p such that

$$H^{\alpha+p+1}[s] \text{ is convergent.}$$

Taking the H^ν-transforms of $[w] \equiv \{w_n\}$ given by (3), for $\nu = p, p-1,\ldots, 1, 0$ successively, we see that

$$H^{\alpha+\nu}[s] \text{ is convergent } (\nu = p, p-1,\ldots, 0);$$

that is, $H^\alpha[s]$ is convergent.

Proof of (b). After (a), we have to prove the result only for the case $c = 0$. Letting $c = 0$ in (3), (4) we have that

$$(H^0 - H^1)H^k[s] \text{ converges (to } l_1, \text{ say).} \tag{6}$$

Since both $H^k[s]$ and $H^{k+1}[s]$ are A-summable, it follows from the properties (ii), (II) of A that $l_1 = 0$ in (6), which we can now write as

$$(H^{-1} - H^0)H^{k+1}[s] = \{o(1)\},$$

i.e. $n(v_n - v_{n-1}) = o(1)$, where $\{v_n\} = v = H^{k+1}[s]$. Since $\{v_n\}$ is A-summable, it follows from (III) that

$$v = H^{k+1}[s] \text{ is convergent.}$$

The argument after (5) shows again that $H^\alpha[s]$ is convergent, and (b) is proved.

Proof of (c). In view of (a) and (b), only the case $c \neq 0$, $\operatorname{re} c = 0$ remains to be proved. Let now $c \neq 0$. We may suppose, in (4), that $\lim w_n = l_0 = 0$, for otherwise we have only to consider $\{s'_n\} \equiv \{s_n + l_0/c\}$ instead of $\{s_n\}$. Following the notation used by Hardy in the proof of his generalization of Mercer's theorem [(**1**) Theorem 52] and replacing 's_n' there by 's_n^*', where $\{s_n^*\} \equiv s^* = H^k[s]$, we have that $\phi_n = o(\log n)$, whence

$$u_n = o(n \log n),$$

so that $$s_n^* = u_n - u_{n-1} = (1+c)n^{-1}u_n + o(1) = o(\log n).$$

It follows from (IV) satisfied by A and the A-summability of $H^k[s]$ in (4) that $H^k[s]$ is convergent.

Using again the argument after (5) in the proof of (a), we see that $H^\alpha[s]$ is convergent, and thus (c) is proved. This completes the proof of the theorem.

Remarks. (1) *Abel*'s method of summability satisfies (i)–(iii), (I), (II), and (III), but is not known to satisfy (IV). *Borel*'s method (B) satisfies

(i)–(iii) and all of (I)–(IV) also. A difficulty in any given case of a summability method is in verifying proposition (III) or (IV).

(2) *Lambert's* method (L) also satisfies (i)–(iii) and (I)–(III) [(II) follows from a result of Rajagopal's (**3**), and (III) follows from (**1**) Theorem 233]. The results (a), (b) of Theorem 1 can thus be seen to be true for the method (L), without making use of the much deeper result that (L)-summability implies Abel-summability [(**1**) Theorem 260].

3. The methods (B), (H,μ_n), and (H^*,μ_{n+1})

Theorem 2 below improves Ramanujan's inclusion theorem [(**4**) Theorem 8] for these methods in the case of bounded sequences.

THEOREM 2. (i) *The conservative methods* (H,μ_n) *and* (H^*,μ_{n+1}) *will sum every Borel-summable bounded sequence* $\sigma \equiv \{\sigma_n\}$ *if*

$$\lim \mu_n = 0, \text{ or equivalently, } \chi(1) = \chi(1-0),$$

where $\chi(t)$ *is a function of bounded variation in* $(0, 1)$ *generating the moment sequence* $\{\mu_n\}$. *The value of the limit is then given by*

$$l = [\chi(+0)-\chi(0)]\lambda+[\chi(1)-\chi(+0)](B)\text{-}\lim \sigma, \tag{7}$$

where $\lambda = \sigma_0$ *or* 0 *according as the method is Hausdorff or quasi-Hausdorff.*

(ii) *If* $\lim \mu_n \neq 0$, *i.e. if* $\chi(1) \neq \chi(1-0)$, *then the conservative methods* (H,μ_n) *and* (H^*,μ_{n+1}) *will sum no Borel-summable bounded divergent sequences.*

For regular methods, part (i) has been proved by Ramanujan (loc. cit.) who gives also an indirect proof of a weaker form of part (ii) [(**4**) 210, remark on Theorem 8].

Proof of Theorem 2. Let $\tau \equiv \{\tau_n\}$, $\tau^* \equiv \{\tau_n^*\}$ denote respectively the transforms of the Borel-summable bounded sequence $\sigma \equiv \{\sigma_n\}$ by the conservative methods (H,μ_n), (H^*,μ_{n+1}). Then τ_n and τ_n^* are each of the form

$$\sum_{k=0}^{\infty} \int_0^1 f_{nk}(t)\sigma_k \, d\chi(t) \quad (n \geqslant 1), \tag{8}$$

where

$$f_{nk}(t) = \binom{n}{k} t^k(1-t)^{n-k} \text{ or } 0 \text{ according as } k \leqslant n \text{ or } k > n, \tag{8a}$$

when (8) gives τ_n, and

$$f_{nk}(t) = \binom{k}{n} t^{n+1}(1-t)^{k-n} \text{ or } 0 \text{ according as } k \geqslant n \text{ or } k < n, \tag{8b}$$

when (8) gives τ_n^* [we may assume in (8b) that $\chi(+0) = \chi(0)$].

Now, by the boundedness of $\{\sigma_n\}$ and the conditions on $\chi(t)$, we can invert the order of operations in (8) and write (8) as

$$\int_0^1 \sum f_{nk}(t)\sigma_k \, d\chi(t) = \int_0^1 F_n(t;\sigma)\, d\chi(t), \tag{9}$$

where $\{F_n(t;\sigma)\}$ is the transform of $\{\sigma_n\}$ by the Euler–Knopp method $E(t)$, or the circle-method $T(t)$, according as we are considering $\{\tau_n\}$ or $\{\tau_n^*\}$. Also

$$|F_n(t,\sigma)| \leqslant \sup|\sigma_n| \quad (0 \leqslant t \leqslant 1\,;\, n \geqslant 0),$$

and $\lim_{t\to+0} F_n(t;\sigma) = \sigma_0$ in (8a), $\lim_{t\to1-0} F_n(t;\sigma) = \sigma_n$ in (8a) and (8b)

Writing

$$\int_0^1 F_n(t;\sigma)\, d\chi(t) = \int_0^\delta \ldots + \int_\delta^{1-\delta} \ldots + \int_{1-\delta}^1 \ldots \quad (0<\delta<1)$$

and keeping n fixed and letting $\delta \to 0$, we see that the expression in (9) is equal to

$$[\chi(+0)-\chi(0)]\lambda + \int_{+0}^{1-0} F_n(t;\sigma)\, d\chi(t) + [\chi(1)-\chi(1-0)]\sigma_n. \tag{10}$$

The methods $E(t)$, $T(t)$ are equivalent, for $0<t<1$, to the Borel-method (B), for bounded sequences. Since $\{\sigma_n\}$ is bounded and Borel-summable, it follows that

$$\lim_{n\to\infty} F_n(t;\sigma) = (B)\text{-}\lim \sigma$$

for each t in $(0,1)$, and hence the middle term in (10) tends to the limit

$$[\chi(1-0)-\chi(+0)](B)\text{-}\lim\sigma$$

as $n \to \infty$.

If $\chi(1) = \chi(1-0)$, then (10) shows that $\{\tau_n\}$, $\{\tau_n^*\}$ converge to the limit l given in (7), and part (i) of the theorem is proved.

If $\chi(1) \neq \chi(1-0)$, we see again from (10) and the argument immediately following it, that each of $\{\tau_n\}$, $\{\tau_n^*\}$ can converge only if $\{\sigma_n\}$ is convergent. This completes the proof of the theorem.

Remarks. (1) If (H^*, μ_{n+1}) is conservative, then for every given a, we have that $\{a, \mu_1, \mu_2, \ldots, \ldots\}$ is a moment sequence, generated, say, by the function $\chi_a(t)$ of bounded variation in $(0,1)$. But the expressions $\chi_a(1)-\chi_a(+0)$, and $\chi_a(1)-\chi_a(1-0)$ are independent of a, which justifies the statement of Theorem 2 in the form given. It may also be noted that, though we have denoted the quasi-Hausdorff method as (H^*, μ_{n+1}) for the sake of a certain convenience of language, we could have denoted it also by (H^*, ν_n).

The assumption that $\chi(+0) = \chi(0)$ in (8b) is no loss of generality since τ_n^* does not involve $\chi(0)$ for $n \geqslant 1$. The assumption justifies the form (9) for τ_n^* $(n \geqslant 1)$.

(2) In the case of regular methods,

$$\chi(+0) = \chi(0), \qquad \chi(1) - \chi(+0) = 1.$$

Hence (7) reduces to $l = (B)\text{-}\lim \sigma$.

Strong regularity, Borel property. The methods (B), $E(p)$, and $T(p)$, for $0 < p < 1$, are equivalent for bounded sequences and each of the methods sums all almost convergent sequences (which are necessarily bounded). Hence we can deduce immediately from Theorem 2 the following improvement of theorems due to Lorentz [(4) Lemma 14] and Ramanujan [(4) Theorem 9].

THEOREM 3. *Each of the conservative methods (H, μ_n), (H^*, μ_{n+1}) will either sum all almost convergent sequences, or will sum no such sequence which is not itself convergent, according as $\lim \mu_n$ is zero or non-zero.*

Finally, recalling that the method (B) has the Borel property, viz., that it sums almost all sequences of 0's and 1's to $\frac{1}{2}$, we get from Theorem 2 the following result, of which the part concerning the regular method (H, μ_n) has been given by Hill, and independently by Lorentz, and cited by Ramanujan [(4) 207–8].

THEOREM 4. *The conservative methods (H, μ_n), (H^*, μ_{n+1}) will sum almost all sequences of 0's and 1's if and only if*

$$\lim \mu_n = 0.$$

If the methods are regular, then this condition is equivalent to the Borel property.

My thanks are due to Professor C. T. Rajagopal and the referee for their helpful remarks.

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ON THE MEANS OF AN ENTIRE FUNCTION AND ITS DERIVATIVES

By K. N. SRIVASTAVA (*Allahabad*)

[Received 19 March 1958]

1. LET $f(z)$ be an entire function of order ρ and lower order λ and let

$$\mu_\delta(r,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta, \tag{1.1}$$

$$\mu_\delta(r,f^{(n)}) = \frac{1}{2\pi}\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^\delta \, d\theta, \tag{1.2}$$

where $f^{(n)}(z)$ is the nth derivative of $f(z)$. It is known [(**4**) 192–5] that

$$\limsup_{r\to\infty} \inf \frac{\log\log\mu_\delta(r,f)}{\log r} = \frac{\rho}{\lambda}. \tag{1.3}$$

In this note I prove the theorem:

THEOREM. *For every entire function* $f(z)$

$$\limsup_{r\to\infty} \inf \frac{\log[r\{\mu_\delta(r,f')/\mu_\delta(r,f)\}^{1/\delta}]}{\log r} = \frac{\rho}{\lambda}, \tag{1.4}$$

where $\delta \geqslant 1$.

2. The proof of this theorem is based on the following two lemmas:

LEMMA 1. *For every entire function* $f(z)$

$$\mu_\delta(r,f') \geqslant \mu_\delta(r,f)\left[\frac{\log\mu_\delta(r,f)}{\delta r\log r}\right]^\delta \tag{2.1}$$

for all $r > r_0 = r_0(f)$, *where* r_0 *depends only on* f *and* $\delta \geqslant 1$.

Proof. We have

$$\begin{aligned}\mu_\delta(r,f') &= \frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|^\delta \, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left|\lim_{\epsilon\to 0}\frac{f(re^{i\theta})-f(r(1-\epsilon)e^{i\theta})}{r\epsilon e^{i\theta}}\right|^\delta d\theta \\ &\geqslant \lim_{\epsilon\to 0}\left[\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{|f(re^{i\theta})|-|f(r(1-\epsilon)e^{i\theta})|}{r\epsilon}\right)^\delta d\theta\right].\end{aligned} \tag{2.2}$$

By Minkowski's inequality [(1) 384]

$$\left[\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{|f(re^{i\theta})|-|f(r(1-\epsilon)e^{i\theta})|}{r\epsilon}\right)^\delta d\theta\right]^{1/\delta}$$

$$\geqslant \frac{1}{r\epsilon}\left[\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^\delta\, d\theta\right)^{1/\delta}-\left(\frac{1}{2\pi}\int_0^{2\pi}|f(r(1-\epsilon)e^{i\theta})|^\delta\, d\theta\right)^{1/\delta}\right]. \quad (2.3)$$

Hence $$[\mu_\delta(r,f')]^{1/\delta} \geqslant \lim_{\epsilon\to 0}\frac{\{\mu_\delta(r,f)\}^{1/\delta}-\{\mu_\delta(r(1-\epsilon),f)\}^{1/\delta}}{r\epsilon}. \quad (2.4)$$

It is well known that $$g(r)=\frac{\log\mu_\delta(r,f)}{\log r}$$

is a monotone increasing function, say for $r>r_0(f)$. In fact G. H. Hardy [(2) 269–77, also (3)] has proved that $\log\mu_\delta(r,f)$ is a convex function of $\log r$. Hence, for $r>r_0$,

$$\{\mu_\delta(r,f')\}^{1/\delta} \geqslant \frac{1}{\delta}\{\mu_\delta(r,f)\}^{1/\delta-1}r^{g(r)-1}g(r)=\frac{1}{\delta}\{\mu_\delta(r,f)\}^{1/\delta}\frac{\log\mu_\delta(r,f)}{r\log r}, \quad (2.5)$$

i.e. $$\mu_\delta(r,f') \geqslant \mu_\delta(r,f)\left\{\frac{\log\mu_\delta(r,f)}{\delta r\log r}\right\}^\delta. \quad (2.6)$$

LEMMA 2. *If $f(z)$ is regular for $|z|\leqslant R$ and if $z=re^{i\theta}$, $0\leqslant r<R$, and $\delta\geqslant 1$, then*

$$\mu_\delta(r,f') \leqslant \frac{1}{(2\pi)^{\delta-1}}\mu_\delta\left[r+r\left\{\frac{\log r}{\log\mu_\delta(r,f)}\right\}^{\frac{1}{2}}\right]\left\{\frac{\log\mu_\delta(r,f)}{r\log r}\right\}^\delta \times$$

$$\times\left[1+\left\{\frac{\log r}{\log\mu_\delta(r,f)}\right\}^{\frac{1}{2}}\right]^\delta. \quad (2.7)$$

Proof. By Cauchy's theorem,

$$|f'(z)| \leqslant \frac{1}{2\pi}\frac{R}{(R-r)^2}\int_0^{2\pi}|f(Re^{i\theta})|\, d\theta. \quad (2.8)$$

Also, by Hölder's inequality [(1) 382],

$$|f'(z)|^\delta \leqslant \left(\frac{1}{2\pi}\frac{R}{(R-r)^2}\int_0^{2\pi}|f(Re^{i\theta})|\, d\theta\right)^\delta$$

$$\leqslant \frac{R^\delta}{(2\pi)^{\delta-1}(R-r)^{2\delta}}\frac{1}{2\pi}\int_0^{2\pi}|f(Re^{i\theta})|^\delta\, d\theta$$

$$=\frac{1}{(2\pi)^{\delta-1}}\frac{R^\delta}{(R-r)^{2\delta}}\mu_\delta(R,f). \quad (2.9)$$

Now taking $$R = r+r\left(\frac{\log r}{\log \mu_\delta(r,f)}\right)^{\frac{1}{2}}$$

and integrating the above inequality with respect to θ between the limits 0 and 2π we get the result.

3. *Proof of the theorem.* From (2.1)

$$\lim_{r\to\infty} \sup \inf \frac{\log[r\{\mu_\delta(r,f')/\mu_\delta(r,f)\}^{1/\delta}]}{\log r} \geqslant \frac{\rho}{\lambda},$$

and from (2.7)

$$\lim_{r\to\infty} \sup \inf \frac{\log[r\{\mu_\delta(r,f')/\mu_\delta(r,f)\}^{1/\delta}]}{\log r} \leqslant \frac{\rho}{\lambda}.$$

Hence the theorem is proved.

COROLLARY. *If* $\lambda = 1$ *and*

$$\liminf_{r\to\infty} \frac{N(r)}{r\log r} > 1,$$

where $N(r)$ *denotes the number of zeros of* $f(z)$ *in* $|z| = r$, *then*

$$\mu_\delta(r,f) < \mu_\delta(r,f') < \ldots < \mu_\delta(r,f^{(n)}) < \ldots$$

for all $r > r_0$.

This is because

$$\log \mu_\delta(r,f) = \log\left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta\right)$$

and, since $|f(re^{i\theta})|$ is continuous, we have, by Jensen's theorem,

$$\log \mu_\delta(r,f) \geqslant \frac{\delta}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| \, d\theta = \delta\int_0^r \frac{N(t)}{t}\,dt - \delta\log|f(0)|.$$

Therefore $$\liminf_{r\to\infty} \frac{\log \mu_\delta(r,f)}{\delta r\log r} > 1.$$

Hence, for $r > r_0$,

$$\left\{\frac{\mu_\delta(r,f')}{\mu_\delta(r,f)}\right\}^{1/\delta} \geqslant \left\{\frac{\log \mu_\delta(r,f)}{\delta r\log r}\right\} > 1.$$

Similar inequalities for higher derivatives give the result.

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ON Q-SPACES AND COLLECTIONS OF CLOSED SETS WITH THE COUNTABLE INTERSECTION PROPERTY

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In this paper we consider spaces in which each collection of closed sets with the countable intersection property (c.i.p.) is contained in a collection of closed sets which is maximal with respect to this property. Such a space will be termed an *I-space*. Results concerning Q-spaces are also obtained.

Note 1. A space is *countably compact* if and only if each collection of closed sets with the finite intersection property (f.i.p.) has the c.i.p. Thus, by Zorn's lemma, a countably compact space is an I-space.

Note 2. A Lindelöf space is a space in which each collection of closed sets with the c.i.p. has a non-void intersection. Thus, a Lindelöf space is an I-space.

In Theorem 2, I-spaces and Q-spaces are related, and the existence of normal T_1-spaces which are not I-spaces is demonstrated.

THEOREM 1. *Let $X = \cup_n X_n$, where each X_n is a closed subset of X and is an I-space in the relative topology. Then X is an I-space.*

Proof. Let $\{A_\alpha\}$ be a collection of closed subsets of X which has the c.i.p. Assume that $\{A_\alpha\}$ is closed under countable intersections. For some n_0, the set X_{n_0} intersects each set A_α. Otherwise, the sequence $\{A_{\alpha_n}\}$, where A_{α_n} does not intersect X_n, would have a void intersection. The collection $\{A_\alpha \cap X_{n_0}\}$ has the c.i.p. since $\{A_\alpha\}$ is closed under countable intersections, and each of its members intersects X_{n_0}. Let $\{B_\beta \cap X_{n_0}\} \supset \{A_\alpha \cap X_{n_0}\}$ with $\{B_\beta \cap X_{n_0}\}$ maximal in X_{n_0} with respect to the c.i.p. and B_β closed in X. The collection of all closed subsets of X each of which contains some member of $\{B_\beta \cap X_{n_0}\}$ is maximal in X with respect to the c.i.p. and contains $\{A_\alpha\}$.

Note 3. A slight generalization of Theorem 1 can be made. Instead of $X = \cup_n X_n$, it is sufficient to assume that $X - \cup_n X_n$ is Lindelöf.

Theorem 1 with Notes 1 and 2 shows the existence of I-spaces which

are neither countably compact nor Lindelöf: for example, the union of a countably infinite number of countably compact spaces at least one of which is not Lindelöf.

Hewitt calls X a 'Q-space' if every maximal free ideal of $C(X, R)$ is hyper-real [(**2**) 85]. $C(X, R)$ is the ring of continuous functions on X to the reals R.

The following is a restatement of a result due to Hewitt [(**2**) Theorem 50].

Note 4. X is a Q-space if and only if each collection $Z(M)$, where M is a maximal ideal in $C(X, R)$, with the c.i.p. has a non-void intersection. $Z(M)$ is the collection of sets of zeros of elements of M.

THEOREM 2. *A normal I-Q-space is Lindelöf.*

Proof. Let $\{A_\alpha\}$ be a collection of closed sets maximal with respect to the c.i.p. Let $\{Z(f_\beta)\}$ be the collection of all zero sets such that each f_β is zero on some member of $\{A_\alpha\}$. The collection $\{f_\beta\}$ is an ideal in $C(X, R)$ and is maximal in $C(X, R)$ if and only if $\{Z(f_\beta)\}$ is maximal with respect to the c.i.p. If $\{Z(f_\beta)\}$ is not maximal, then there exists a zero set Z which intersects each member of $\{Z(f_\beta)\}$ but is not a member of $\{Z(f_\beta)\}$. Thus, Z is not in $\{A_\alpha\}$ and, since this collection is maximal with respect to the c.i.p., it contains a set A such that $Z \cap A = 0$. But, since X is normal, there is a zero set Z_0 which contains A and does not intersect Z. This contradicts the assumption that Z intersects each member of $\{Z(f_\beta)\}$ since, by the definition of this collection, Z_0 is a member of it. Since X is a Q-space, by Note 4, $\cap_\beta Z(f_\beta) \neq 0$, and since a closed set in a completely regular space is the intersection of the zero sets containing it, $\cap_\beta Z(f_\beta) = \cap_\alpha A_\alpha$. This shows that each maximal c.i.p. collection of closed sets has a non-void intersection. In an I-space this is clearly equivalent to the Lindelöf property.

The condition of normality in Theorem 2 can be relaxed somewhat. It is easy to see that it is sufficient to be able to separate, by disjoint zero sets, pairs of disjoint closed sets Z, A, where Z is a zero set and both sets are uncountable. The uncountable part follows from the fact that, in any c.i.p. collection of closed sets, each member must be uncountable unless the intersection of the entire collection is non-void. For an example of a completely regular I-space in which the only pairs of disjoint closed sets which cannot be separated by open sets contain one countable set see [(**2**) 69]. By the theorem such a space is not a Q-space since it would be Lindelöf and thus normal. In this connexion we recall [(**2**) Theorem 19] that, if X is a T_1-space in which each pair of

disjoint closed sets can be separated by a pair of disjoint zero sets, then X is normal.

There are normal Q-spaces which are not Lindelöf [(**2**) 87]. As a matter of fact, Bing [(**1**) 361] has shown the existence of normal Q-spaces which are not paracompact, and so not Lindelöf. Such spaces are not I-spaces by Theorem 2. Particularly, an uncountable discrete space with non-measurable cardinal is a Q-space [(**1**) 352] which is not Lindelöf, and hence not an I-space. This is equivalent to the statement that an uncountable set with non-measurable cardinal contains a collection of subsets which has the c.i.p. and is not contained in a maximal c.i.p. collection. This presents the following problem: *construct a c.i.p. collection of subsets of some uncountable set which is not contained in a maximal c.i.p. collection.*

THEOREM 3. *A T_1-I-space X can be densely embedded in a Lindelöf space by applying Wallman's compactification method* (3), *where the points of the embedding space are the maximal c.i.p. collections of closed subsets of X. A completely regular T_1-space X can be densely embedded in a Q-space, where the points of the embedding space are the collections of zero sets of maximal ideals in $C(X, R)$ which are maximal with respect to the c.i.p. This Q-space is homeomorphic to $\nu(X)$, the subspace of $\beta(X)$ on which all bounded continuous functions can be extended to $\beta(X)$.*

The proof is straightforward.

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ON TWO MANUSCRIPTS BY BISHOP BARNES

By W. N. BAILEY (*Eastbourne*)

[Received 7 November 1958]

1. At the end of his classical memoir 'On generalized Legendre functions', published in 1907, E. W. Barnes (**3**) stated 'To be continued'. In 1954, when I was writing an obituary notice of Barnes for the London Mathematical Society, Professor G. N. Watson suggested that, in view of the note at the end of the 1907 memoir, I should ask Mrs. Barnes if her husband had left any unpublished manuscripts. At the time only one manuscript could be found, and this gave the formula

$$F_4[\alpha,\beta;\gamma,\alpha+\beta-\gamma+1;z(1-Z),Z(1-z)] = F(\alpha,\beta;\gamma;z)F(\alpha,\beta;\alpha+\beta-\gamma+1;Z), \quad (1.1)$$

valid inside simply-connected regions surrounding $z=0$, $Z=0$ for which

$$|z(1-Z)|^{\frac{1}{2}}+|Z(1-z)|^{\frac{1}{2}}<1.$$

This formula may be regarded as giving a case when Appell's function F_4 can be expressed in a simple form, or as giving an expansion of the product of two hypergeometric functions with different variables.

The formula (1.1) was obtained by myself (**1** and **2**) in 1933, and I did not know until the discovery of the manuscript that Barnes had obtained the formula a quarter of a century before I did. The manuscript was quite a lengthy one and gave different forms of (1.1) with a number of diagrams illustrating the regions of validity in the different cases. I sent the work to Professor Watson, and at one time he contemplated writing a short note on it, but was deterred by the fact that the state of printing at the time was very difficult.

2. About six months ago I heard again from Mrs. Barnes. She had been tidying up some of her father's papers (not on mathematics) and she had found another manuscript by her husband. She kindly sent me the work, and I found that it was certainly a continuation of the memoir of 1907. Like the previous manuscript it considers products of functions with different variables.

This manuscript begins with a reference to a paper by Bateman (**4**),

who had proved the formula

$$\frac{1}{\{\Gamma(n+1)\}^2}\left(\frac{x_1+x_2}{2}\right)^n P_n\left\{\frac{x_1x_2+1}{x_1+x_2}\right\} = \sum_{t=0}^{n} \frac{2t+1}{\Gamma(t+n+2)\Gamma(n-t+1)} P_t(x_1)P_t(x_2), \tag{2.1}$$

and the more general result

$$\left(\frac{x_1+x_2}{2}\right)^n P_n^m\left\{\frac{x_1x_2+1}{x_1+x_2}\right\}\frac{1}{n!(m+n)!}$$

$$= \sum_{t=0}^{n} \frac{(2t+1)\Gamma(t+1-m)}{\Gamma(t+m+1)\Gamma(n-t+1)\Gamma(t+n+2)} P_t^m(x_1)P_t^m(x_2), \tag{2.2}$$

where m and n are positive integers.

Barnes remarks that these results are simple cases of a large class of analogous formulae. The manuscript is a very lengthy one and is not nearly prepared for publication. I therefore suggested to Mrs. Barnes that I should write a note giving the main results of the manuscript without proofs, and this I propose to do here. The early formulae go in pairs, one giving a product of two functions as an integral, and the other being obtained from this by summation and the evaluation of an integral of Barnes' type by residues.

3. The first formula proved is

$$P_t^m(x_1)P_t^{-m}(x_2)$$

$$= -\frac{1}{2\pi i}\int \frac{\Gamma(t+1+s)}{\Gamma(t+1-s)}\frac{\Gamma(-s)}{\Gamma(1+s-m)}\left(\frac{x_2-x_1}{2}\right)^s P_s^{-m}\left\{\frac{x_1x_2-1}{x_1-x_2}\right\} ds, \tag{3.1}$$

where the integral embraces the positive half of the real axis and encloses solely the positive sequence of poles of the subject of integration. From this is deduced the formula

$$\left(\frac{x_2-x_1}{2}\right)^n P_n^m\left\{\frac{x_1x_2-1}{x_1-x_2}\right\}$$

$$= n!\,\Gamma(1+m+n)\sum_{t=0}^{n} \frac{(-1)^t(2t+1)}{\Gamma(t+n+2)\Gamma(n-t+1)} P_t^{-m}(x_1)P_t^m(x_2), \tag{3.2}$$

where n is a positive integer and x_1, x_2, $(x_1x_2-1)/(x_1-x_2)$ have their real parts positive and are not less than or equal to unity; the formula is true for general complex values of m.

It is noted that, when $m = 0$, (3.2) reduces to (2.1) since

$$P_t(-x_1) = (-1)^t P_t(x_1)$$

when t is a positive integer. Also, by a similar method, Barnes proves the formula

$$\frac{1}{n!}\left(\frac{x_1+x_2}{2}\right)^n P_n\left\{\frac{x_1x_2+1}{x_1+x_2}\right\} = \sum_{t=0}^{n} \frac{1}{t!(n-t)!}\left(\frac{x_1-x_2}{2}\right)^t P_t\left\{\frac{x_1x_2-1}{x_1-x_2}\right\}. \quad (3.3)$$

Barnes now remarks 'We have now to try to discover the more general theorem of which the result of the present paragraph is a special case: many tentative investigations led to the following theory'.

4. He then proceeds to prove that

$$\Gamma(-n)\Gamma(n+1)\Gamma(1+n-m)\Gamma(-n-m)P_n^m(x_1)P_n^m(x_2)$$
$$= -\frac{1}{2\pi i}\int \Gamma(-s)\Gamma(-m-s)\Gamma(s-n)\Gamma(s+n+1)\times$$
$$\times\left(\frac{x_1+x_2}{2}\right)^s P_s^m\left\{\frac{x_1x_2+1}{x_1+x_2}\right\} ds, \quad (4.1)$$

where x_1, x_2, $(x_1x_2+1)/(x_1+x_2)$ have their real parts positive and are not less than or equal to unity.

In the proof of this he first proves that

$$\Gamma(\alpha)\Gamma(\beta)\Gamma(\rho_1-\alpha)\Gamma(\rho_1-\beta)F\{\alpha,\beta;\rho_1;y_1\}F\{\alpha,\beta;\rho_1;y_2\}$$
$$= -\frac{1}{2\pi i}\Gamma(\rho_1)\int \Gamma(-s)\Gamma(1-\rho_2-s)\Gamma(s+\alpha)\Gamma(s+\beta)(1-y_1)^s(1-y_2)^s\times$$
$$\times F\left\{1-\rho_2-s,-s;\rho_1;\frac{y_1y_2}{(1-y_1)(1-y_2)}\right\} ds,$$

where $\alpha+\beta=\rho_1+\rho_2-1$, $|\arg(1-y_1)|$ and $|\arg(1-y_2)|$ are both less than π, and $|y_1y_2/(1-y_1)(1-y_2)|<1$. He then deduces (4.1) as a particular case.

The next formula to be obtained is

$$(2/\pi)\sin(n+1)\pi\,\Gamma(-n-m)\Gamma(n+1-m)Q_n^m(x_1)P_n^m(x_2)$$
$$= -\frac{1}{2\pi i}\int \frac{\Gamma(s)\Gamma(s-m)\Gamma(n+1-s)}{\Gamma(n+1+s)}\left(\frac{x_1+x_2}{2}\right)^{-s} e^{\pi i(s-n-1)}\times$$
$$\times P_{-s}^m\left\{\frac{x_1x_2+1}{x_1+x_2}\right\} ds \quad (4.2)$$

and this may equally be written as

$$(2/\pi)\sin(n+1)\pi\,Q_n^{-m}(x_1)P_n^m(x_2)$$
$$= -\frac{1}{2\pi i}\int \frac{\Gamma(s)\Gamma(n+1-s)}{\Gamma(n+1+s)\Gamma(1+m-s)}\left(\frac{x_1+x_2}{2}\right)^{-s} P_{-s}^m\left\{\frac{x_1x_2+1}{x_1+x_2}\right\} ds. \quad (4.3)$$

From (4.3) Barnes now deduces the expansion

$$\frac{\Gamma(-n)\pi}{2\Gamma(1+m+n)\sin n\pi}\left(\frac{x_1+x_2}{2}\right)^n P_n^m\left\{\frac{x_1x_2+1}{x_1+x_2}\right\}$$

$$=\sum_{t=0}^{\infty}\frac{(-1)^{t-1}(2t-2n-1)\Gamma(t-2n-1)}{t!}Q_{t-n-1}^{-m}(x_1)P_{t-n-1}^{m}(x_2), \quad (4.4)$$

where x_1, x_2, $(x_1x_2+1)/(x_1+x_2)$ have their real parts positive and are not less than or equal to 1, and $|\xi_1/\xi_2|<1$, where

$$\xi = x-\sqrt{(x^2-1)};$$

the formula is then true for complex values of n, m, x_1, x_2.

5. The next result obtained is that, for general values of n and q,

$$(X^2-1)^{-\frac{1}{2}q}Q_n^q(X) = \sqrt{(2\pi)}(x_1^2-1)^{-\frac{1}{2}q}(x_2^2-1)^{-\frac{1}{2}q}(z^2-1)^{\frac{1}{2}-\frac{1}{2}q}\times$$

$$\times\sum_{m=0}^{\infty}\frac{(m+q)\Gamma(m+2q)}{m!}Q_n^{m+q}(x_2)P_n^{-m-q}(x_1)P_{m+q-\frac{1}{2}}^{\frac{1}{2}-q}(z), \quad (5.1)$$

where $X = x_1x_2+z\sqrt{\{(x_1^2-1)(x_2^2-1)\}}$. Then, taking $z=1$ in this formula, it is shown that (4.4) is a particular case.

Special cases of (4.4) noted are

$$\frac{1}{x_1+x_2}=\sum_{t=0}^{\infty}(-1)^t(2t+1)P_t(x_2)Q_t(x_1), \quad (5.2)$$

true for all values of x_1 and x_2 if $|\xi_1|<|\xi_2|$, and

$$\sum_{t=0}^{\infty}\frac{(-1)^{t-1}(2t+2q+1)\Gamma(t+2q+1)}{t!}Q_{t+q}^{q}(x_1)P_{t+q}^{-q}(x_2)$$

$$=\frac{\pi}{\sin q\pi\,\Gamma(-2q-1)}(x_1+x_2)^{-2q-1}(x_1^2-1)^{\frac{1}{2}q}(x_2^2-1)^{\frac{1}{2}q}. \quad (5.3)$$

It is also deduced that, if n is a positive integer, then for general complex values of m

$$\frac{1}{n!\,\Gamma(m+n+1)}\left(\frac{x_1+x_2}{2}\right)^n P_n^m\left\{\frac{x_1x_2+1}{x_1+x_2}\right\}$$

$$=\sum_{t=0}^{n}\frac{(2t+1)(-1)^t}{\Gamma(n-t+1)\Gamma(n+t+2)}P_t^m(x_2)P_t^{-m}(-x_1). \quad (5.4)$$

Another formula is

$$\frac{\pi\Gamma(-n)}{2\sin n\pi\,\Gamma(m+n+1)}\left(\frac{x_1+x_2}{2}\right)^n Q_n^m\left\{\frac{x_1x_2+1}{x_1+x_2}\right\}$$
$$=\sum_{t=0}^{\infty}(-1)^{t-1}\frac{(2t-2n-1)\Gamma(t-2n-1)}{t!}Q_{t-n-1}^{-m}(x_1)Q_{n-t}^m(x_2), \quad (5.5)$$

provided that x_1, x_2, $(x_1x_2+1)/(x_1+x_2)$ have their real parts positive and are not less than or equal to 1, and $|\xi_1/\xi_2|<1$.

6. A more general result is now obtained for hypergeometric series, viz.

$$(1+wy_1)^{-\alpha}(1+wy_2)^{-\beta}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}F\left\{\alpha,\beta;\ \alpha+\beta;\ \frac{-w(1-y_1)(1-y_2)}{(1+wy_1)(1+wy_2)}\right\}$$
$$=\sum_{m=0}^{\infty}\frac{\Gamma(\alpha+m)\Gamma(\beta+m)}{m!\,\Gamma(\alpha+\beta+m)}(-w)^mF\{\alpha,-m;\ 1-\beta-m;\ y_1\}\times$$
$$\times F\{\beta,-m;\ 1-\alpha-m;\ y_2\} \quad (6.1)$$

if $|wy_1y_2|<1$, $|w|<1$, and neither α nor β has a real positive value which makes a series on the right infinite, and provided that y_1, y_2 do not lie on the cross-cuts of the functions involved.

After discussing the additon formulae for $Q_n(X)$ and $P_n(X)$, the manuscript gives the formula

$$(X^2-1)^{-\frac{1}{4}q}P_n^{-q}(X)=\sqrt{(2\pi)}(x_1^2-1)^{-\frac{1}{4}q}(x_2^2-1)^{-\frac{1}{4}q}(z^2-1)^{\frac{1}{4}-\frac{1}{4}q}\times$$
$$\times\sum_{m=0}^{\infty}\frac{(m+q)\Gamma(1+m+q+n)\Gamma(m+2q)\Gamma(1+n-q)}{\Gamma(1+n+q)\Gamma(1+m)\Gamma(1+n-q-m)}\times$$
$$\times P_n^{-m-q}(x_1)P_n^{-m-q}(x_2)P_{m+q-\frac{1}{2}}^{\frac{1}{2}-q}(z), \quad (6.2)$$

where as before $X=x_1x_2+z\sqrt{\{(x_1^2-1)(x_2^2-1)\}}$, and the formula is true for general complex values of n and q.

The rest of the manuscript is very obscure, but I think that I have given enough to indicate briefly what it contains.

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1. W. N. Bailey, *Quart. J. of Math.* (Oxford) (1) **4** (1933) 305–8.
2. —— ibid. 5 (1934) 291–2.
3. E. W. Barnes, *Quart. J. of Pure and App. Math.* **39** (1908) 97–204.
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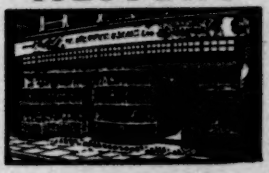

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